THE SIXTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Sixth Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Thursday, December 28, 1899. Beside the usual election of officers and the presentation of annual reports, the occasion was especially marked by the presidential address of President R. S. Woodward, entitled "The century's progress in applied mathematics." An invitation to attend the reading of the address had been extended to the American Physical Society, which was present in a body, the total attendance reaching about eighty persons, of whom the Mathematical Society furnished nearly one-half. President H. A. Rowland of the Physical Society was placed in the chair during the joint session, which was extended to include the reading of the paper by Professor Pupin, noted below. The event afforded a pleasant renewal of the cordial relations of the two scientific bodies and a continued assurance of their hearty coöperation.

The following thirty-eight members of the Society were registered as in attendance at the two sessions:

Dr. E. M. Blake, Professor Maxime Bôcher, Dr. W. G. Bullard, Professor J. E. Clark, Dr. A. Cohen, Professor F. N. Cole, Dr. W. S. Dennett, Dr. F. Durell, Professor A. M. Ely, Professor T. S. Fiske, Mr. A. S. Gale, Miss Carrie Hammerslough, Dr. G. W. Hill, Dr. A. A. Himowich, Professor Harold Jacoby, Mr. S. A. Joffe, Professor Pomeroy Ladue, Professor P. A. Lambert, Professor Gustave Legras, Dr. Emory McClintock, Dr. James Maclay, Dr. Isabel Maddison, Professor Mansfield Merriman, Dr. G. A. Miller, Dr. D. A. Murray, Professor G. D. Olds, Mr. J. C. Pfister, Professor James Pierpont, Professor M. I. Pupin, Professor J. K. Rees, Mr. C. H. Rockwell, Dr. Virgil Snyder, Professor Henry Taber, Miss Mary Underhill, Professor J. H. Van Amringe, Professor L. A. Wait, Professor A. G. Webster, Professor R. S. Woodward.

The President of the Society, Professor R. S. Woodward, occupied the chair during the separate sessions of the Society. The Council announced the election of the following persons to membership: Professor William Beebe, Yale University, New Haven, Conn.; Dr. J. V. Collins, State Normal School, Stevens Point, Wis.; Professor A. R. Forsyth, Trinity College, Cambridge, England; Professor M. W. Haskell, Uni-

versity of California, Berkeley, Cal.; Dr. Emilie N. Martin, Philadelphia, Pa.; Mr. C. A. Noble, University of California, Berkeley, Cal.; Mr. E. B. Wilson, Yale University, New Haven, Conn.; Miss R. G. Wood, New Haven, Conn. Four applications for membership were reported.

Reports were received from the Treasurer, the Librarian, and the Auditing Committee. These reports will be printed in the Annual Register, now in preparation. The Secretary reported that the total number of members of the Society was now 337, including twelve life members. Thirty-six new members joined the Society during the year 1899. The total attendance of members at the meetings during the year was 197, and the number of members attending at least one meeting was 110. The number of papers presented was 106, as against 88 in 1898.

On the occasion of the retirement of Professor Harold Jacoby from the position of Treasurer, appropriate resolutions were adopted expressing recognition of his valuable services, extending with a brief interruption from the founding of the Society.

At the annual election the following officers and members of the Council were chosen :

President,	Professor R. S. WOODWARD.
First Vice-President,	Professor E. H. MOORE.
Second Vice-President,	Professor T. S. FISKE.
Secretary,	Professor F. N. COLE.
Treasurer,	Dr. W. S. DENNETT.
Librarian,	Professor POMEROY LADUE.

Committee of Publication,

Professor F. N. COLE,
Professor ALEXANDER ZIWET,
Professor FRANK MORLEY.

Members of the Council to serve until December, 1902,

Professor OSKAR BOLZA,
Professor SIMON NEWCOMB,
Professor L. A. WAIT.

The following papers were presented :

(1) Dr. G. A. MILLER : "On the groups which have the same group of isomorphisms."

(2) Professor MAXIME BÔCHER: "On regular singular points of linear differential equations of the second order whose coefficients are not necessarily analytic."

(3) Dr. VIRGIL SNYDER: "On cyclical quartic surfaces in space of n dimensions."

(4) Dr. VIRGIL SNYDER: "On the geometry of the circle."

(5) Mr. W. B. FITE: "A proof that the commutator subgroup of a group may contain operators which are not commutators."

(6) J. E. CAMPBELL, M.A.: "On the types of linear partial differential equations of the second order (in three independent variables) which are unaltered by the transformations of a continuous group."

(7) Professor L. E. DICKSON: "Proof of the existence of the Galois field of order p^r for every integer r and prime number p."

(8) Dr. E. M. BLAKE: "On plane movements whose point loci are of order not greater than four."

(9) Professor R. S. WOODWARD: Presidential address, "The century's progress in applied mathematics."

(10) Professor M. I. PUPIN: "The propagation of electrical waves over non-uniform conductors."

(11) Professor HENRY TABER: "On the singular transformations of groups generated by infinitesimal transformations."

(12) Dr J. I. HUTCHINSON: "On certain relations among the theta constants."

(13) Professor E. O. LOVETT: "Singular solutions of Monge equations."

Mr. Fite's paper was presented to the Society through Dr. G. A. Miller, and Mr. Campbell's paper through the Secretary. In the absence of the authors, Mr. Fite's paper was read by Dr. Miller, Dr. Hutchinson's by Dr. Virgil Snyder, and those of Mr. Campbell, Professor Dickson, and Professor Lovett were read by title.

The presidential address appeared in the January number of the BULLETIN (pp. 133–163), and has also been published in *Science*. vol. 11 (new series), no. 263, pp. 41–51; no. 264, pp. 81–92. The papers of Professor Dickson, Dr. Snyder, and Professor Taber are contained in the present number of the BULLETIN. Mr Campbell's paper will be published in the *Transactions*. Abstracts of the remaining papers are given below.

The main object of Dr. Miller's paper was the determination of all the possible groups whose group of isomorphisms

or whose group of cogredient isomorphisms is the symmetric group of order six. It was shown that these groups are the direct products of abelian groups and an infinite system of groups which contains just one group for every power of two and is thus analogous to the system of Hamiltonian groups whose order is a power of two.

The following theorems were also proved: If a group is generated by two characteristic subgroups that have only identity in common, its group of isomorphisms is the direct product of the groups of isomorphisms of these two characteristic subgroups. The group of isomorphisms of an abelian group A is abelian whenever A is cyclical, and it is non-abelian when A is non-cyclical. The necessary and sufficient condition that a cyclical group of order n is the group of isomorphisms of some group is that n is of the form $p^{\alpha}(p-1)$, p being an odd prime number. If the group of cogredient isomorphisms G_1 of a group G transforms one of its operators A_1 of order p^{α}, p being any prime number, into its k^{th} power and if $k \not\equiv 1 \mod p$, then at least one operator of order p^{α} corresponds to A_1 in the isomorphism between G and G_1; if $k \equiv 1 \mod p^{\gamma}$, but $\not\equiv 1 \mod p^{\gamma+1}$, then at least one operator of order $p^{\alpha+\gamma'}$ ($\gamma' \equiv \gamma$) corresponds to A_1 in the isomorphism between G and G_1.

In Professor Bôcher's paper, which has now appeared in the *Transactions* (Volume 1, number 1, pp. 40–52), the questions which the author considered on pp. 280–281 of the last volume of the BULLETIN, are discussed by another method by means of which it is possible to treat the subject much more exhaustively.

In the geometry of the circle, as discussed by Dr. Snyder, the circle is taken for generating element analogous to the sphere in Lie's higher spherical geometry. A linear equation between the five coördinates defines the ∞^2 circles which cut a fixed circle at a constant angle. Several problems in construction were easily solved, *e. g.*, to find a circle cutting three given ones at given angles. The elements of differential geometry were given, and the differential equation of all envelopes of circles expressed. In this geometry a quadratic complex gives the bicircular quartics when the point circles form one coördinate complex, and a new system of circular curves of order 24 when the coördinates are unrestricted. Some important properties of both systems of curves were obtained.

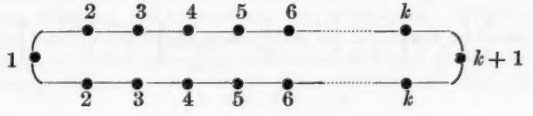
Mr. Fite's paper was in abstract as follows: Let G be a group of order p^m, p being a prime number, which has an abelian group of cogredient isomorphisms, generated by four independent operators P, Q, R, S; and let A, B, C, D be the operators of G that correspond to P, Q, R, S, respectively. The commutator subgroup of G will contain operators which are not commutators of G whenever the following conditions are satisfied:

$$B^{-1}AB = h_1A, \quad C^{-1}AC = h_2A, \quad D^{-1}AD = h_3A,$$
$$C^{-1}BC = h_4B, \quad D^{-1}BD = h_5B, \quad D^{-1}CD = h_6C,$$

where h_1, h_2, $\cdots$, h_6 are independent self-conjugate operators of G different from identity. There is a group of order 1024, obtained by establishing a certain isomorphism for G_3^4 written in six distinct systems of elements, which satisfies all of the given conditions. There is also a group of order 256 whose commutator subgroup contains an operator which is not a commutator.

Dr. Blake considered the movements of a rigid body or system one of whose planes slides upon a fixed plane and is further restricted to those movements which cause all points of the system to generate curves of order not greater than four. A list of all known movements of this character is given in Part I. and the problem of a complete enumeration is proposed. The locus of a straight line oblique to the plane of movement may be a scroll of order either not greater than four or greater than four. The author has discussed these scrolls for all known movements of the former kind. They are the movements of the ellipsograph (article by the author in manuscript); movements whose centrodes are congruent conics (*American Journal of Mathematics*, vol. 21, p. 257); the degenerate case of three-bar movement whose centrodes are limaçons; and Roberts's generalization of the conchoid mechanism of Nicomedes. The author's present paper treats the last two cases in Parts II. and III.

In Professor Pupin's paper the conductor is a long wire having coils 1, 2, $\cdots$, k, $k+1$ interposed at equidistant points, as in the figure.



An electromotive force of type Ee^{ipt} is impressed at point 1. The problem discussed leads to the integration of the differential equation of a plane wave in an absorbing medium, namely,

$$L\frac{dx^2}{dt^2}+R\frac{dx}{dt}=\frac{1}{C}\frac{d^2x}{ds^2},$$

where s is the distance of any point from point 1, x is the current at that point, t is the time, and L, R, C are the inductance, ohmic resistance, and capacity, respectively, of the wire per unit length.

The integral has to satisfy $k+2$ boundary conditions, one at each coil. To formulate these, let $x_1, x_2, \cdots, x_{k_2}, x_{(k+1)_2}$ be the currents at points $1, 2, \cdots, k, k+1$. Also let L_0, R_0, C_0 be the inductance, capacity, and resistance, respectively, of any one of the $k+2$ equal coils. Then at point 1

$$\frac{\delta x_1}{\delta s}=\frac{1}{2}ipCEe^{ipt}-\frac{C}{2}(-p^2L_0+ipR_0)x_1=\frac{1}{2}D_0-\frac{H}{2}x_1;$$

at point $k+1$.

$$\frac{\delta x_{(k+1)_2}}{\delta s}=-\frac{H}{2}x_{(k+1)_2};$$

and at any other point a

$$\left(\frac{\delta x_{a_2}}{\delta s}\right)_1-\left(\frac{\delta x_{a_2}}{\delta s}\right)_2=-Hx_a,$$

where a may be any integer from 1 to k.

The integral does not seem to be obtainable by ordinary methods. The method of successive eliminations which the author developed in his paper on "Electrical oscillations in a loaded conductor," read at the meeting of the Society, February 25, 1899, succeeds in finding an integral.

Thus the current in the section between the points a and $a+1$ at a distance ξ from a is given by

$$x_{a_2+\xi}=\frac{[\frac{1}{2}h_0+h_0']\cos\left(\frac{al}{k+1}+\xi\right)\mu-d\cos\left\{\frac{(k+1-a)l}{k+1}-\xi\right\}\mu}{\mu\sin\mu l}$$

$$+\frac{h_0}{\mu}\left\{x_{a_2}\sin\left[\frac{(a-1)l}{k+1}+\xi\right]\mu+x_{2_2}\sin\left[\frac{(a-2)l}{k+1}+\xi\right]\mu\right.$$

$$\left.+\cdots+x_{a_2}\sin\mu\xi\right\},$$

where

$$h_0 = C(-p^2 L_0 + ipR_0), \quad -\mu^2 = C(-p^2 L + ipR),$$

$$h_0' = h_0 \left\{ x_s \cos \frac{k\mu l}{k+1} + x_{2s} \cos \frac{(k-1)l\mu}{k+1} + \cdots + x_{ks} \cos \frac{l\mu}{k+1} \right\}$$

The currents $x_1, x_2, \cdots x_{ks}, x_{(k+1)s}$ are determined from the following $k+2$ simultaneous equations:

$$\begin{aligned}
&(\sigma_1 + 1)\, x_1 - x_s = \frac{\rho}{2} + \frac{\sigma_1}{2} x_1, \\
&(\sigma_1 + 2)\, x_s - x_{2s} - x_1 = 0, \\
&(\sigma_1 + 2)\, x_{2s} - x_{3s} - x_{2s} = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&(\sigma_1 + 2)\, x_{ks} - x_{(k+1)s} - x_{(k-1)s} = 0, \\
&(\sigma_1 + 1)\, x_{(k+1)s} - x_{ks} = + \frac{\sigma_1}{2} x_{(k+1)s},
\end{aligned}$$

where

$$\sigma_1 = \frac{h_0 \sin \dfrac{\mu l}{k+1}}{\mu} - 4 \sin^2 \tfrac{1}{2} \frac{\mu l}{k+1}, \qquad \rho = \frac{\tfrac{1}{2} pcEe^{ipt} \sin \dfrac{\mu l}{k+1}}{\mu}$$

By putting $\sigma_1 = -4 \sin^2 \psi$, we obtain any current

$$x_{ms} = -\frac{\dfrac{\rho}{2} \cos 2(k - m + 2)\, \psi}{\sin 2\psi \sin (2k+2)\, \psi}.$$

This disposes of the waves of forced period. The waves of free period can be obtained by putting

$$\sin 2\psi \sin(2k+2)\psi = 0, \qquad \therefore\ (2k+2)\psi = \nu\pi,$$

where ν may be any integer from 0 to $2k+2$.

The theta constants considered by Dr. Hutchinson are the values of the theta functions with half integer characteristics for zero arguments. Relations among the thetas of two different Göpel systems are considered. From these can be derived the expression of all the theta constants in terms of those belonging to the same Göpel group (or system). This is the direct generalization of the results

already well known for the hyperelliptic theta functions (depending upon two arguments).

Professor Lovett's paper, which is intended for publication in the *Transactions*, employs Lie's theory of infinitesimal transformations to construct a method for determining the singular solutions of Monge and Pfaff equations.

F. N. COLE.

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THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE Sixth Semi-Annual Meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held on December 28 and 29, 1899, at the University of Chicago. The following members of the Society were in attendance:

Professor Oskar Bolza, Professor E. W. Davis, Professor Thomas F. Holgate, Dr. Kurt Laves, Professor H. Maschke, Professor John A. Miller, Professor E. H. Moore, Professor Alexander Pell, Professor D. A. Rothrock, Professor G. T. Sellew, Professor E. B. Skinner, Dr. H. E. Slaught, Dr. H. F. Stecker, Professor C. A. Waldo, Dr. J. V. Westfall, Professor H. S. White, Professor Mary F. Winston, Professor J. W. A. Young.

Professor E. H. Moore, Vice-President of the Society, occupied the chair during the first of the four sessions, after which Professor E. W. Davis presided. The Christmas meeting being the regular time for the election of officers of the Section, the Secretary was re-elected and Professors H. B. Newson and C. A. Waldo were elected members of the programme committee. The time and place of the next meeting were fixed for Saturday, April 14, 1900, at Northwestern University, Evanston, Ill.

The following papers were presented:

(1) Mr. R. E. MORITZ: "A generalization of the process of differentiation."

(2) Professor E. D. ROE: "On the transcendental form of the resultant."

(3) Dr. E. J. WILCZYNSKI: "An application of Lie's theory to hydrodynamics."

(4) Mr. F. R. MOULTON: "On the question of the stability of certain particular solutions of the problem of four bodies, together with particular solutions of the problem of n bodies of the Lagrangian type.

(5) Professor L. E. DICKSON: "The canonical form of linear homogeneous substitutions in a general Galois field."

(6) Professor L. E. DICKSON: "The cyclic subgroup of the simple group of linear fractional substitutions of determinant unity in two non-homogeneous variables with coefficients in an arbitrary Galois field."

(7) DR. J. V. WESTFALL: "On a category of transformation groups in space of four dimensions."

(8) Professor OSKAR BOLZA: "The elliptic sigma functions considered as a special case of the hyperelliptic sigma functions."

(9) Professor ALEXANDER PELL: "Calculation of the integral

$$\int_0^\infty e^{-\left(px^2+\frac{q}{x^2}\right)} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \left\{ rx^2 + \frac{s}{x^2} \right\} dx."$$

(10) Professor JOHN A. MILLER: "Concerning certain modular functions of square rank."

(11) Professor R. J. ALEY: "A new collinear set of three points connected with a triangle."

(12) Professor H. MASCHKE: "Note on the unilateral surface of Moebius."

(13) Professor C. A. WALDO: "On a family of warped surfaces connected by a simple functional relation."

(14) Professor H. S. WHITE: "Plane cubics and irrational covariant cubics."

Mr. Moritz's paper was presented to the Society through Professor Davis, and was read by him in the absence of the author. Mr. Moulton's paper was presented through Dr. Laves. In the absence of the authors, Professor Roe's paper was read by Professor White, Dr. Wilczynski's by Professor Bolza, Professor Dickson's by Professor Moore, and Professor Aley's by the Secretary.

After the regular papers had been presented, a general discussion upon the topic "Limits of functions of one or more variables" was opened by Professor Moore.

Mr. Moritz's paper was a contribution to the doctrine of pure forms. Let $\overset{+}{}_n$ denote the symbol of a combinatory process of the nth order, $\overset{-}{}_n$ and $\underset{-}{}_n$ its two inverses, so that if $a \overset{+}{}_n b = c$, then $c \overset{-}{}_n a = b$, and $c \underset{-}{}_n b = a$; and let $a \overset{+}{}_n M_n = a$, for

all values of a, define M_n, the modulus of the process. The process of the $(n+1)$th order is built up from the process of the nth order, as multiplication is built up from addition. Three general theorems were proved: I. If M_n is the modulus of a process of the nth order, then $M_{n+1}\bar{}_1 M_n$ is indeterminate. II. If any process possesses a modulus, and is subject to the sub-associative law, viz.:

$$a_n^+ b_n^+ c = a_n^+ (b_{n-1}^+ c),$$

then

$$a \bar{}_n b = (a \bar{}_n c)_{n-1}\bar{}(b \bar{}_n c).$$

III. If in addition to the hypothesis of theorem II, the process is subject to the sub-distributive law, viz.:

$$a_{n+1}^+ (b_{n-1}^+ c) = (a_{n+1}^+ b)_n^+ (a_{n+1}^+ c)$$

then

$$M_{n+1}\bar{}_1 a = M_{n-1}.$$

The symbol Y_n was defined by the equation

$$Y_n = {}^{\lim}_{h=M_n}\{f(x_n^+ h)\bar{}_n f(x_{n+1}\bar{}_1 h)\};$$

for $n=1$ this becomes $Y_1 = \frac{dy}{dx}$. For any process subject to the conditions of theorem I., Y_n is indeterminate. If, moreover, the conditions of theorem II. and III. exist for the process in question, then by theorem II.

$$Y_n = (M_{n+1}\bar{}_1 a)_n (M_{n+1}\bar{}_1 a)$$

and by theorem III. $Y_n = M_{n-1}\bar{}_n M_{n-1}$, that is, it will be possible to evaluate Y_n provided Y_{n-1} can be evaluated. For $n=2$ all the above conditions exist, and we have in fact

$$Y_2 = {}^{\lim}_{h=1}\left\{\log_h \frac{f(xh)}{f(x)}\right\}.$$

The symbol $\frac{dy}{dx}$ was used to denote the forms thus defined, and the term quotiential coefficient was applied to them. A table of quotiential coefficients accompanied the paper.

Professor Roe's paper found its origin in two letters of Gordan, one to Hermite (*Comptes Rendus*, 127 (1898), p. 539), the other to the author, and arose from an attempt to remove certain difficulties suggested in the former of these

and to work out the formulas contained in the latter. Three functions

$$f = x^m + a_1 x^{m-1} + \cdots + a_m,$$
$$\varphi = x^n + b_1 x^{n-1} + \cdots + b_n,$$
$$\phi = 1 + b_1 x + b_2 x^2 + \cdots + b_n x^n,$$

where the roots of ϕ are the reciprocals of those of φ, are assumed, and the transcendental form of the resultant of f and ϕ is obtained as

$$R = \sum_{\nu=0}^{\nu=\infty} \Sigma \frac{(-1)^{\rho_1+\rho_2+\cdots}}{\lambda_1^{\rho_1}\lambda_2^{\rho_2}\cdots\,\rho_1!\,\rho_2!\cdots} \{S_{\lambda_1}(a)\cdot S_{\lambda_1}(b)\}^{\rho_1} \{S_{\lambda_2}(a)\cdot S_{\lambda_2}(b)\}^{\rho_2}\cdots$$

where $\rho_1 + \rho_2 + \cdots = \nu$ and $S_\lambda(a)$, $S_\lambda(b)$ are respectively the sums of the nth powers of the roots of f and φ. It is seen, however, that R is also the resultant of f and ϕ for any other degrees than m and n and the conclusion is reached: *The transcendental form expresses all binary resultants formally in the same form and is a universal formula comprehending them all. It is the most general formula for binary resultants.*

The finite form of the resultant $R_{m,n}$ is reached by dropping all the terms of R of weight greater than mn which, when expressed in the a's and b's for given functions f and ϕ, are identically zero. We have,

$$R_{m,n} = \Sigma\Sigma \frac{(-1)^{\rho_1+\rho_2+\cdots}}{\lambda_1^{\rho_1}\lambda_2^{\rho_2}\cdots\,\rho_1!\,\rho_2!\cdots} S_\lambda(a) S_\lambda(b),$$

with $\lambda_1\rho_1 + \lambda_2\rho_2 + \cdots \leqq mn$. This $R_{m,n}$ is still very general, formally representing all resultants $R_{\mu,\nu}$, where $\mu\nu = mn$, in the same form, until the a's and b's for given equations are substituted, when it becomes definite. It may also represent expressions which are not resultants.

A one parameter group can, as Lie has noticed, be interpreted as representing the steady motion of a fluid. In Dr. Wilczynski's paper, Lie's suggestion was followed out, a number of general theorems being thus obtained. The theory was then applied to the special cases which from this standpoint are the simplest and most important, viz., that the groups are projective, linear, or linearoid. If the forces which act upon the unit of mass have a potential, the most general ternary projective group which can represent a

steady fluid motion is the general linear group. This motion was studied in detail, one of the results being that, if the coördinate planes are appropriately chosen, the projection of the orbit of every point of the fluid upon the plane of xy is a conic section. All of these conic sections are similar and similarly placed.

If the motion is periodic, and also in other cases, the orbits themselves are conics in parallel planes; in the periodic case, of course, ellipses. A special case is the ellipsoid whose particles describe such elliptic orbits, a problem treated by Dirichlet and Dedekind. Interesting results were also found for the more difficult problem when the group is linearoid. The paper will be offered for publication to the *Transactions*.

At the Columbus meeting of the Society, August, 1899, a paper was presented by Mr. Moulton exhibiting twenty-eight particular solutions of the problem of four bodies, under the limitation that one of them should be infinitesimal. The first part of his present paper discussed the question of the stability of the motion of the infinitesimal body. Stability was defined as follows: The particular solutions have been already defined by those constant values of the coördinates and projections of the velocity of the infinitesimal body, satisfying the differential equations of motion. It is now supposed that the initial conditions are slightly different from those of the exact solutions,

$$x_0,\quad y_0,\quad z_0,\quad \frac{dx_0}{dt},\quad \frac{dy_0}{dt},\quad \frac{dz_0}{dt}.$$

Suppose $x = x_0 + x'$, etc., at the origin of time, where x', y', etc., are small quantities in terms of which the differential equations are now developed. All terms of degree higher than the first in the new variables are neglected, and the resulting linear system solved. If the roots of the *characteristic equation* are all pure imaginaries the generating solution is said to be *stable;* otherwise it is said to be *unstable.*

In the case where all four of the bodies lie in a line two of the roots of the characteristic equation are found to be real and two pure imaginary for all masses of the finite bodies. In the case where the infinitesimal body does not lie in the same line with the others, the roots of the characteristic equation are all pure imaginaries, or all complex, depending upon the relative masses of the finite bodies. In the case where the finite bodies are equal in mass and are

at the vertices of an equilateral triangle, the roots of the characteristic equation are all pure imaginaries.

The second part of the paper took up particular solutions of the problem of n bodies of the Lagrangian type. The Lagrangian solutions of the problem of three bodies may be defined, from the standpoint of their derivation, as those solutions in which the ratios of the mutual distances of the bodies are constants; or, from the standpoint of their other properties, as those solutions in which each of the three bodies describes a conic section in the same period, and in such a manner that the center of gravity is at one of the foci of each conic, and that the law of areas holds for each body considered separately. Adopting the second definition, the necessary and sufficient conditions for solutions of this type of the problem of n bodies were written in algebraic form. It was proved that for any given n bodies there are $\frac{1}{2}n!$ solutions in which the bodies all lie in a line. There are doubtless solutions in which the bodies do not all lie in a line, but they are not proved to exist in the general case. In any particular example where the masses are given there is no difficulty in finding solutions if they exist. The associated problem, given n arbitrary points on a line, to find n masses such that, if they are placed at these points, solutions of the Lagrangian type will exist was also solved.

A more extended paper on this subject by Mr. Moulton, of which the present one will form a part, will be published in the *Transactions*.

The result due to Jordan (Traité des Substitutions, pp. 114–126) on the canonical form of a linear substitution on m variables with integral coefficients taken modulo p, a prime, may be readily generalized to substitutions in the $GF[p^n]$. Instead of following Jordan's method of proof, Professor Dickson in his first paper gave a simpler proof by induction. The theorem was supposed to hold for substitutions in the field which have the characteristic determinant (in the parameter K)

$$[F_k(K)]^{\alpha-1}[F_l(K)]^{\beta}\cdots \quad (F_k, F_l \cdots \text{irreducible})$$

and was then proved to hold for every substitution in the field, which has the characteristic determinant

$$F_k^{\alpha}F_l^{\beta}\cdots \qquad (m \equiv k\alpha + l\beta + \cdots).$$

If we proceed by induction from $m-1$ to m indices, we

fail to prove that part of the theorem concerning the conjugacy of the new indices. The author constructs explicitly the general substitution in the $GF[p^n]$ which is commutative with any given substitution in the field. In particular the number of the former substitutions was determined and found to agree with that derived by a different analysis by Jordan (Traité, p. 136) for the case $n = 1$.

Professor Dickson's second paper leads to a revision and a generalization to the $GF[p^n]$ of certain results due to Professor Burnside (*Proceedings of the London Mathematical Society*, vol. 26, pp. 58–106) upon groups of substitutions with integral coefficients taken modulo p, a prime. Numerous variations from Burnside's method of treatment were introduced, partly to avoid the separation of the two cases $p \equiv 1$ and $p \equiv -1 (\mathrm{mod}\ 3)$ and to include the cases $p = 2$ and $p = 3$, and partly to minimize the calculations by the frequent use of known general theorems.

Burnside's results are incorrect in two places. The factor 2 should be deleted from $\frac{N}{2(p-1)^2}$ on pages 103 and 104; in fact, the statements made on page 103, lines 15–23, do not lead to the conclusion stated. Of substitutions having the canonical form

$$x' = x, \quad y' = y + x, \quad z' = z + y,$$

there exist three conjugate sets of N/p^2 substitutions, instead of one conjugate set of $3N/p^2$ substitutions as stated by Burnside, p 102, lines 1–6.

The generalization leads to an interesting result in the particular case $p^n = 2^2$. We have then a simple group of order 20160. The distribution of its cyclic subgroups in sets of conjugates is given by the following table:

960	conj.	cyclic	groups	of	order	7	with	5760	subs.	period 7
2016	"	"	"	"	"	5	"	8064	"	" 5
630	"	"	"	"	"	4	"	1260	"	" 4
630	"	"	"	"	"	4	"	1260	"	" 4
630	"	"	"	"	"	4	"	1260	"	" 4
315	"	"	"	"	"	2	"	315	"	" 2
1120	"	"	"	"	"	3	"	2240	"	" 3
								1	"	" 1

Total number of distinct substitutions 20160.

It follows at once that this group is not isomorphic with the alternating group on 8 letters having the order 20160. In fact, the latter contains substitutions of periods 6 and 15.

The two groups contain the same number of substitutions of period 7, the same number of periods 4 and 2. This non-isomorphism was first proved by Miss Schottenfels under the direction of Professor Moore. The corresponding investigation for three non-homogenous variables is now being made by Mr. T. M. Putnam.

Both of Professor Dickson's papers will appear in the *American Journal of Mathematics*.

Lie, in an article in the *Leipziger Berichte* for October, 1899, proceeding from Helmholtz's assumptions, omitting one, namely, the monodrom axiom, has determined all the groups, satisfying the given conditions under the most general interpretation of the axiom, of "unrestricted motion" (freie Beweglichkeit). He finds, besides the groups of euclidian and non-euclidian motion, five others. Kowalewski, a pupil of Lie, in his inaugural dissertation, has extended Lie's investigations in space of three dimensions to that of four and five. Besides the euclidian and non-euclidian groups, he finds in space of four dimensions three others A, B, and C. From analogy with the groups determined by Lie, he makes the statement, without proof, that group A satisfies the monodrom axiom, while B and C do not. By suitably choosing the points of general position, which are to be held fixed, and by considering the invariants of these points, Dr. Westfall proved that A satisfies the monodrom axiom, without being compelled to resort to the long integration otherwise necessary. In the case of group B, he showed that, contrary to Kowalewski's statement, the group may or may not satisfy this axiom, according to the mutual situation of the points held fixed.

To prove that A always satisfies the monodrom axiom he proceeds as follows: 1. Hold the special triplet of general points

$$P_1 = (0, 0, 0, 0),\ P_2 = (0, 0, z_3, z_4),\ P_3 = (0, 0, z_3', 0)$$

fixed and obtain the subgroup $(x_1 p_2 - x_2 p_1)$ which has closed path curves. 2. The three invariants of this point triplet are positive and never equal to zero. They are, further, independent with respect to z_3, z_4, and z_3'. If we choose any real positive values whatever, and set them equal to these invariants, we can solve for z_3, z_4, and z_3'. 3. From the form of the general invariant of the group we see that all triplets of points of general position have positive invariants which can never be equal to zero. If, therefore, we choose any triplet of points whatever, we can

always determine z_3, z_4, and z_3' so that the invariants of the one triplet will be equal to the invariants of the other, and we can, therefore, transform one into the other by means of a transformation of the group. In the special case the path curves were closed, and we therefore know they are closed in general.

By suitable choice of the points to be held fixed and by consideration of the invariants in the case of group B, the following results were proved: 1. All groups obtained by holding fixed a triplet of general points whose invariants are all negative have closed path curves. 2. These are all the subgroups that have this characteristic.

Professor Bolza's paper gave a sketch of the theory of the elliptic sigma functions as they appear in the light of the theory of the hyperelliptic sigma functions. Section 1 contains systems of associated integrals of the first, second, and third kinds; section 2, Weierstrass's theta functions, (a) as functions of u, (b) in the Riemann surfaces; section 3, the Al-functions and the function $p(u)$; section 4, the sigma functions and the functions $\wp u$ and their invariantive properties; section 5, the partial differential equation of the sigma functions and the recursion formulæ for their expansion into power series. The paper has now been published in the *Transactions* (Vol. 1, No. 1, pp. 53–65).

Professor Miller's paper dealt with the properties of certain modular elliptic functions, the rank of which is a square number. The major part of the discussion was given to the particular cases of functions whose ranks are 9 and 4 respectively. Certain theorems used in the treatment of these special cases are true when the rank is any square number, as was proved. In the course of the discussion two groups were found; the one a quaternary group of linear substitutions of order 648, the other a ternary group of order 192. The letters of the substitutions of both these groups, the elements of the groups, and the operational character of the elements, were derived from the consideration of the X-functions, Z-functions, and $\sigma_{\lambda,\mu}$-functions, defined in many places, chiefly by Professor Klein. In case the rank $n = 9$ it is possible to define $t_\alpha (\alpha = 1, \cdots, 4)$ as linear homogeneous functions of Z_α, and thereby to find a group of monomial substitutions on t_α which is holoedrically isomorphic with the group G_{648} referred to above. It was shown that there are an infinite number of forms in t_α, which are invariant under G_{648}, and in ac-

cordance with the Hilbert law the basis of this system of forms was found. The function $\sigma_{\lambda/3,\,\mu/3}(w_1, w_2)$ was obtained as a linear homogeneous function of t_α, and by means of this the basis of the infinite system was expressed as rational functions of g_2, g_3, and Δ, where g_2, g_3, and Δ have the signification usually given them in the treatment of elliptic functions.

The treatment of modular functions the rank of which is four differs in some of its details from that of the preceding, because the definition of $X_\alpha(u)$ in case n is even differs slightly from that of $X_\alpha(u)$ in case n is odd. However, the ends to be reached in the two cases are nearly identical. In both cases an interesting linear homogeneous connection has been found between

$$\sigma_{\frac{\lambda}{n},\,\frac{\mu}{n}}(nu, w_1, w_2) \quad \text{and} \quad X_{\frac{\alpha}{n^2}}(u, w_1, w_2).$$

In Professor Aley's paper it was pointed out that the isogonal conjugate of the Brianchon point of any triangle, the center of the inscribed circle, and the center of the circumscribed circle are collinear.

Professor Maschke explained briefly the connection between the well known Moebius unilateral paper strip and the ruled surfaces of the third order, illustrating his demonstration by a model. A short account of the paper has been published in the *Transactions* (Volume 1, number 1, p. 39).

Professor Waldo's paper contained a demonstration of the following theorem: The equation

$$f\left(\frac{px}{p-z}, \frac{qy}{q-z}\right) = 0$$

represents a skew surface with two rectilinear directors whose equations are

$$(y = 0, \quad z = p) \quad \text{and} \quad (x = 0, \quad z = q),$$

and a plane curve director lying in the XY plane, having for its equation $f(x, y) = 0$, the common perpendicular to the two rectilinear directors being chosen for Z-axis. By this theorem the equations of the whole family of skew surfaces having two distinct rectilinear directors can be at once written down and their principal deforma-

tions shown under cartesian form. The surfaces having for curvilinear directors $x - y = 0$ and $xy - \frac{1}{2} = 0$ were studied in detail and models exhibited showing their principal types.

Professor White's paper was a further development of the topic considered in the paper presented by him at the Columbus meeting of the Society. Each mixed concomitant (2, 2) of the cubic defines (as in the paper referred to) two covariant nets of conics. These are polars of two cubics of the syzygetic sheaf; the totality of such is exactly that entire sheaf of cubics. But these concomitants (2, 2) and all the concomitants (3, 3) serve to define also *four* covariant sheaves of cubics, not in the syzygetic sheaf, intimately connected on the one hand with the four inflexional triangles, and on the other hand with the eighteen collineations of the cubic into itself. This paper will be published in the *Transactions*.

THOMAS F. HOLGATE.
Secretary of the Section.

EVANSTON, ILL.

ON CYCLICAL QUARTIC SURFACES IN SPACE OF *N* DIMENSIONS.

BY DR. VIRGIL SNYDER.

(Read before the American Mathematical Society, December 28, 1899.)

THE generation of the cyclide as the envelope of spheres which cut a fixed sphere orthogonally and whose centers lie on a quadric can readily be generalized to space of n dimensions.

In ordinary space it appears that the same surface is the envelope of five different systems; that the quadric loci of centers are all confocal and the associated spheres are all orthogonal; that the possibilities of the system are exactly coextensive with the ∞^{13} possible cyclides.

Let

$$(1) \quad (x_1 - ix_{n+2}) \sum_{r=-}^{n} y_r^2 - 2 \sum_{r=1}^{n} x_{r+1} y_i + (x_1 + ix_{n+2}) = 0$$

be the equation of a sphere in R_n; it contains $n + 2$ homo-

geneous constants x_r, and if the radius be denoted by

$$\frac{ix_{n+3}}{x_1 - ix_{n+2}},$$

the quadratic identity

$$x \equiv \sum_{r=1}^{n+3} x_r^2 \equiv 0 \tag{2}$$

will exist among these $n+3$ numbers x_r, which may be called the homogeneous *coördinates of the sphere.*

Two spheres a, b will intersect orthogonally when

$$\sum_{r=1}^{n+3} a_r b_r = 0,$$

the terms defining the radii of the two spheres not occurring; hence any linear equation of the form

$$\sum_{r=1}^{n+2} a_r x_r = 0 \tag{3}$$

represents the ∞^n spheres which cut a fixed sphere orthogonally.

Now consider a quadratic equation of the form

$$\varphi_2(x_1, x_2, \cdots, x_{n+2}) = 0 \tag{4}$$

which does not contain x_{n+3}, and make it simultaneous with (3); between the two $x_1 + ix_{n+2}$ may be eliminated, leaving a quadratic equation among the point coördinates of the centers of the variable spheres.

Hence, equations (3) and (4) define the ∞^{n-1} spheres which cut a fixed sphere orthogonally, and whose centers lie on a quadric surface M_{n-1}^2. These spheres envelop a new surface whose equation may be found as follows:

Let $x_1 - ix_{n+2}$ be replaced by x_1, as $x_1 + ix_{n+2}$ has been eliminated between (3) and (4). Similarly, let $x_1 + ix_{n+3}$ be eliminated between (3) and (1). Then, with a slight change in the meaning of the coefficients, the problem reduces to that of finding the envelope of the sphere

$$s \equiv x_1 [a_{n+2}(y_1^2 + y_2^2 + \cdots + y_n^2) - a_1] - 2 \sum_{r=1}^{n} x_{r+1} (a_{n+2} y_r - a_{r+1}) = 0,$$

subject to the condition, say $f(x_1, x_2, \cdots, x_{n+1}) = 0$.

Since the sphere s is to touch the envelope f, it may be regarded as the equation of the point of contact, and f as the equation of the surface itself, in tangential coördinates, hence

$$\Sigma \frac{\partial f}{\partial x_i} x_i = \Sigma \frac{\partial s}{\partial x_i} x_i = 0,$$

and corresponding coefficients must be proportional,

$$\frac{\partial f}{\partial x_r} = \lambda \frac{\partial s}{\partial x_r}.$$

Between these $n+1$ linear equations in x_r and the equation $s = 0$ the numbers x_r and λ may be eliminated, giving

$$(a) \qquad \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} & a_{n+1} \sum_{r=1}^{n} y_r^2 - a_1 \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} & a_2 - 2y_1 a_{n+2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_3} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_3 \partial x_n} & a_3 - 2y_2 a_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n+2} \sum_{1}^{n} y_r^2 - a_1 & a_2 - 2y_1 a_{n+2} & \cdots & \cdots & 0 \end{vmatrix} = 0$$

or the Hessian of f bordered by the coefficients of x_r in s.

This proves that the envelope is a cyclical surface, and the number of constants in the most general surface of this kind, viz., $\frac{1}{2}(n^2 + 5n + 2)$, exactly coincides with the number of constants in f, $\frac{1}{2}n(n+3)$, plus the number in (3), $n+1$, so that all quartic cyclical surfaces can be generated in this way. This does not show, however, in how many ways the same surface may be generated.

Let the two forms

$$x, \quad \varphi_2 + a_n x_{n+2}^2$$

be subjected to any linear transformation, such that $x = 0$ may go into itself, and φ_2 may become a sum of squares of the form

$$(5) \qquad F \equiv \sum_{r=1}^{n+2} b_r x_r^2 = 0;$$

further, suppose the restriction be also imposed that x_{n+3} shall go into itself. Equation (5) represents a complex of spheres; in the vicinity of any of its spheres z it may be replaced by the tangent linear complex

$$\sum_{r=1}^{n+3} \frac{\partial F}{\partial z_r} x_r = 0,$$

so that all of the spheres which belong to F and touch z must touch it in points of its $M_{n-2}^{2,2}$ of intersection with the fundamental sphere of the tangent complex. When

$$\sum_{r=1}^{n+3} \left(\frac{\partial F}{\partial z_r}\right)^2 = 0,$$

these two spheres touch each other. The $M_{n-2}^{2,2}$ reduces to a point, and z is a singular sphere.

Let $\partial F/\partial z = t_r$; now t is also a sphere which touches z, and the whole tangent pencil can be represented in the form

$$m_r = t_r + \lambda z_r = b z_r + \lambda z_r = z_1 (b_r + \lambda).$$

The sphere m touches z, which is also a sphere, hence

$$\sum_{r=1}^{n+3} m_r x_r = 0, \qquad \sum_{r=1}^{n+3} z_r^2 = 0,$$

or, replacing z_r by its values,

$$A \equiv \Sigma \frac{m_r^2}{b_r + \lambda} = 0, \qquad \Sigma \frac{m_r^2}{(b_r + \lambda)^2} = 0.$$

Those values of m which satisfy these two equations define the singular sphere of a quadratic complex for every value of λ, hence the whole pencil of complexes have the same surface of singularities. The original complex is contained in the series, corresponding to $\lambda = \infty$.

The λ eliminant of these two equations will give the equation in tangential coördinates of the surface of singularities; the surface is seen to be of class $4n$.

Among the quadratic complexes of the pencil are $n + 3$ simple ones counted twice, corresponding to $\lambda = -b_r$. When $\lambda = -b_{n+3}$ this becomes the complex of points in R_n which, combined with $x = 0$ and with the other terms of the complex

$$\sum_{r=1}^{n+2} \frac{x_r^2}{b_r - b_{n+3}} = 0,$$

defines a general cyclical surface. But the envelopes of the other $n+2$ systems corresponding to $\lambda = -b_r$ ($r = 1, \cdots, n+2$) define the same surface. Now $x_r = 0$ represents the totality of spheres which cut a fixed sphere orthogonally, and when associated with $A = 0$ defines all those spheres whose centers also lie on a quadric, hence by (a) they envelop a cyclical surface. The fundamental spheres of these $n+2$ systems are by (3) mutually orthogonal, and the quadrics are confocal since $A = 0$. Hence

An M_{n-1}^4 in (euclidean) R_n which contains the absolute as a double $M_{n-2}^{1,2}$ can be generated in $n+2$ ways as the envelope of those hyperspheres which cut a fixed hypersphere orthogonally and whose centers lie on a $M_{n-3}^{2,3}$. The fixed spheres are mutually orthogonal and the quadrics are confocal.

The $M_{n-3}^{2,2}$, intersections of the M_{n-1}^2 and the hyperspheres, are all focal spreads of the cyclical surface. Through the center of each sphere passes a bitangent cone M_{n-1}^2 whose elements are perpendicular to the elements of the asymptotic, cones of the M_{n-1}^2.

By giving λ different values in the system

$$\sum_{r=1}^{n+3} \frac{x_r^2}{a_r - \lambda} = 0, \qquad x_{n+3} = 0,$$

a series of confocal cyclical surfaces is obtained. By substituting the coördinates of a point sphere in the equation, n different values for λ can be found; hence, *n cyclical surfaces of a confocal system pass through every point in space;* by applying the tangent complex to each and using (3) it appears that *these cyclical surfaces intersect orthogonally.*

For $n = 2$ these surfaces (bicircular quartic curves) have been systematically studied, from a different point of view, by Casey, Darboux, Cox, Loria, and others; for $n = 3$ (cyclides) by Casey, Maxwell, Cayley, Darboux, Reye, Loria, Bôcher, Domsch, Loewy, Moutard; and many special points have been noticed in numerous other papers.

The method here given is a generalization of that first employed by Darboux, using Lie's more general coördinates. The latter were first systematically employed in an article by the author, read at the Toronto meeting of the AMERICAN MATHEMATICAL SOCIETY, and published in the BULLETIN, volume 4 (new series), pp. 144–154.

For $n = 4$, the number of distinct types is 58, and for larger values of n the number of types has not been determined.

CORNELL UNIVERSITY,
November 2, 1899.

ON THE SINGULAR TRANSFORMATIONS OF GROUPS GENERATED BY INFINITESIMAL TRANSFORMATIONS.

BY PROFESSOR HENRY TABER.

(Read before the American Mathematical Society, December 28, 1899.)

By means of r independent infinitesimal transformations

$$X_j = \sum_1^n \xi_{ji}(x_1, x_2, \cdots, x_n) \frac{\partial}{\partial x_i} \qquad (j = 1, 2, \cdots, r)$$

we may construct a family of transformations

$$(1) \qquad x_i' = x_i + \sum_1^r a_j X_j x_i + \tfrac{1}{2} \sum_1^r \sum_1^r a_j a_k X_j X_k x_i + \cdots$$
$$\equiv f_i(x_1, \cdots, x_n, a_1, \cdots, a_r) \qquad (i = 1, 2, \cdots, n)$$

with r essential parameters $a_1, a_2, \cdots, a_r$. The transformations defined by these equations, for assigned values of the a's, may be denoted by T_a. Each transformation of this family is paired with its inverse.

For finite values of the parameters a, the transformation T_a (provided it is not illusory) belongs to a one parameter group generated by the infinitesimal transformation

$$a_1 X_1 + \cdots + a_r X_r.$$

As the a's approach certain limiting values, one or more of which is infinite, T_a may have a definite finite transformation T as a limit. The transformation T may be regarded as a transformation of the family, and, if equivalent to a transformation T_b with finite parameters, can be generated by an infinitesimal transformation of the family (namely, $b_1 X_1 + \cdots + b_r X_r$), but not otherwise.*

Let it be assumed that

$$X_j X_k - X_k X_j = \sum_1^r c_{jks} X_s \qquad (j, k = 1, 2, \cdots, r),$$

*Thus, if the transformation T_a, for one or more of the a's infinite, is finite and definite, but is not equivalent to a transformation of the family with finite parameters, the transformation T_a cannot be generated by an infinitesimal transformation of the family. To this extent Lie's theorem on p. 65 of the Transformationsgruppen, vol. 1, requires modification.

the c's being constants. Then, by the chief theorem of Lie's theory, the family of transformations (1) forms a group G with continuous parameters; and each transformation of G is, in general, generated by an infinitesimal transformation of the group. Thus from

$$(1) \qquad x_i' = f_i(x_1, \cdots, x_n, \ a_1, \cdots, a_r) \qquad (i = 1, 2, \cdots, n)$$

and

$$(2) \qquad x_i'' = f_i(x_1', \cdots, x_n', \ b_1, \cdots, b_r) \quad (i = 1, 2, \cdots, n)$$

we derive

$$(3) \qquad x_i'' = f_i(x_1, \cdots, x_n, \ c_1, \cdots, c_r) \qquad (i = 1, 2, \cdots, n)$$

where

$$(4) \qquad c_j = \varphi_j(a_1, \cdots, a_r, \ b_1, \cdots, b_r) \quad (j = 1, 2, \cdots, r).$$

For finite values of the a's and b's it may happen that every branch of one or more of the functions φ is infinite. In this case, while each of the transformations T_a and T_b is generated by an infinitesimal transformation of the group, the transformation T_bT_a, resulting from their composition, cannot be generated thus, and the group cannot properly be said to be continuous. A transformation of G which cannot be generated by an infinitesimal transformation of G may be termed *essentially singular*.

In what follows I shall signify by T_a, T_b, etc., transformations of the groups with finite parameters generated, respectively, by the infinitesimal transformations

$$a_1X_1 + a_2X_2 + \cdots + a_rX_r, \quad b_1X_1 + b_2X_2 + \cdots + b_rX_r, \quad \text{etc.}$$

Group G may contain a transformation T_a (generated by an infinitesimal transformation) which, in composition with every transformation of some one (or more) subgroups of G with one parameter, in particular with the infinitesimal transformation of such subgroup, results in an essentially singular transformation. Such a transformation T_a I term *non-essentially singular*. The values of the a's for which T_a is non-essentially singular may be termed *critical* values of the parameters. The critical values of the parameters are included among those for which one or more of the roots of the equation in ρ

$$\begin{vmatrix} \sum_1^r a_f c_{f11} - \rho, & \sum_1^r a_f c_{f21} & , \cdots \\ \sum_1^r a_f c_{f12} & , \sum_1^r a_f c_{f22} - \rho, & \cdots \\ \cdots\cdots\cdots\

$$\begin{vmatrix} P_{11}(a), & P_{12}(a), & \cdots \\ P_{21}(a), & P_{22}(a), & \cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{vmatrix}$$

and let $A_{\mu\nu}$ denote the first minor of Δ_a relative to $P_{\nu\mu}(a)$. Then, if we put

$$a_j = \varphi_j(\bar{a}_1, \cdots, \bar{a}_r, tb_1, \cdots, tb_r) \quad (j = 1, 2, \cdots, r),$$

regarding $\bar{a}_1, \cdots, \bar{a}_r, b_1, \cdots, b_r$ as fixed and the a's as functions of t, the latter satisfy the differential equations

$$\Delta_a \frac{da_j}{dt} = A_{j1}b_1 + A_{j2}b_2 + \cdots + A_{jr}b_r \qquad (j = 1, 2, \cdots, r).$$

The A's are integral functions (transcendental or rational) of the a's, and Δ_a vanishes only if one or more of the roots of the characteristic equation of $\mathfrak{F}_a$ is an even multiple, not zero, of $\pi\sqrt{-1}$.*

The functions φ are multivalued. Returning to the original notation, let

$$c_j = \varphi(a_1, \cdots, a_r; b_1, \cdots, b_r) \quad (j = 1, 2, \cdots, r).$$

For assigned values of the a's and b's the difference between any two branches of $\varphi_j(a, b)$ is equal to

$$2\pi\sqrt{-1}\sum_1^r{}_k m_k \psi_{kj}(a, b),$$

where the m's are integers, and the ψ's are rational functions of the coefficients of the bilinear form $e^{\mathfrak{F}_a}e^{\mathfrak{F}_b}$.

The equations defining the transformations of one group may restrict the functions φ_j to fewer branches than in the case of another group of the same structure (*Zusammensetzung.*) Consequently, of two groups of the same structure, one may be continuous and the other may be discontinuous, that is, may possess essentially singular transformations.†

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*If we denote by T_b the transformation of G whose parameters are $tb_1, tb_2, \cdots, tb_r$, which is generated by the infinitesimal transformation $b_1X_1 + \cdots + b_rX_r$, the transformation $T_bT_{\bar{a}}$ is essential singular only for those values of t for which the determinant is zero of the bilinear form $e^{\mathfrak{F}_b}e^{\mathfrak{F}_{\bar{a}}} - I$ / $\mathfrak{F}_b\mathfrak{F}_{\bar{a}}$.

†My attention was first drawn to this fact by my pupil, Mr. S. E. Slocum.

If, however, the adjoined of any group G is discontinuous, G itself, and, of course, every group of the same structure is discontinuous. The bilinear form $\mathfrak{F}_\alpha$ is closely related to the adjoined group. In fact, if ϕ_α denotes the matrix of $\mathfrak{F}_\alpha$, the infinitesimal equations of the adjoined are

$$(a_1', a_2', \cdots, a_r) = (1 + \partial t \phi_\alpha)(a_1, a_2, \cdots, a_r);$$

and we have $$e^{\mathfrak{F}_\beta} e^{\mathfrak{F}_\alpha} = e^{\mathfrak{F}_\gamma}$$

where $$\gamma_j = \varphi_j(a_1, \cdots, a_r, \beta_1, \cdots, \beta_r) \qquad (j = 1, 2, \cdots, r).$$

CLARK UNIVERSITY,
December, 1899.

PROOF OF THE EXISTENCE OF THE GALOIS FIELD OF ORDER p^r FOR EVERY INTEGER r AND PRIME NUMBER p.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, December 28, 1899.)

EXISTENCE proofs have been given by Serret* and by Jordan.† The developments used by Serret are lengthy but quite in the spirit of Kronecker's ideas. The short proof by Jordan, however, assumes with Galois the existence of imaginary roots of an irreducible congruence modulo p.

The proof sketched in this note proceeds by induction. Assuming the existence of the $GF[p^n]$, we derive that of the $GF[p^{nq}]$, q being an arbitrary prime number. Since the $GF[p]$ exists, being the field of integers taken modulo p, it will follow that the $GF[p^q]$ exists, and by a simple induction that the $GF[p^r]$ exists for r arbitrary.

We employ the lemma: *A factor of $x^{p^{nm}} - x$, belonging to and irreducible in the $GF[p^n]$, can be of degree m' if and only if m' divides m.* In particular, the irreducible factors of $x^{p^{nq}} - x$ are of degree q or 1. But the product of the distinct‡ linear factors $x - \nu_i$ belonging to the $GF[p^n]$ is $x^{p^n} - x$.

*Algèbre supérieure, 2, pp. 122–142.

†Traité des substitutions, pp. 16, 17.

‡Two functions belonging to the $GF[p^n]$ are called distinct if one is not the product of the other by a constant, a mark of the field.

Every such factor $x - \nu_i$ *is a simple factor of* $x^{p^{nq}} - x$. The statement being evident for the factor x, we assume $\nu_i \neq 0$, and proceed to prove that $x - \nu_i$ is not a factor in the field of

$$\frac{x^{p^{nq}-1} - 1}{x^{p^n-1} - 1} \equiv (x^{p^n-1})^{r-1} + \cdots + (x^{p^n-1}) + 1,$$

where $$r \equiv (p^{nq} - 1)/(p^n - 1).$$

But for $x = \nu_i \neq 0$, this function reduces in the $GF[p^n]$ to the value r, which is evidently not divisible by p.

It follows that the quotient $(x^{p^{nq}} - x)/(x^{p^n} - x)$ breaks up into factors of degree q belonging to and irreducible in the $GF[p^n]$. Any such factor may be used to define the $GF[p^{nq}]$.

MÉRAY'S INFINITESIMAL ANALYSIS.

Leçons nouvelles sur l'analyse infinitésimale et ses applications géométriques. Par M. CH. MÉRAY, professeur à la Faculté des Sciences de Dijon. Paris, Gauthier-Villars. Première partie: *Principes généraux*, 8vo., 1894, xxxiii + 405 pp.; Deuxième partie: *Étude monographique des fonctions principales d'une seule variable*, 1895, xi + 495 pp.; Troisième partie: *Questions analytiques classiques*, 1897, vi + 206 pp.; Quatrième partie: *Applications géométriques classiques*, 1898, xi + 248 pp.

As a work of art—and mathematics is preëminently a fine art—this monumental work of Méray's is a masterpiece; as a treatise to induct students of mathematics into the mysteries of the infinitesimal analysis under the direction of any other than the author* its success is hardly so unqualified. Constructed so as to require technically as preliminary training nothing but a knowledge of elementary algebra and the theory of systems of linear algebraic equations, the first volume would indicate either or both of two conditions: that the author is a most marvelous teacher, that his pupils are of a race superior to that brilliant body of mathematicians now at Paris. It is one thing to be possessed of suffi-

* The author states in the preface to the first volume (1894) that for twenty-four years he has found his method uniformly satisfactory in teaching.

cient elementary knowledge to undertake the serious study of Méray's treatise and quite another thing for that preliminary training to have developed the power of abstraction necessary to work through it.

Méray seems never to have had a particular idea, except at second thought or second hand. He lisped in generalities, for the general ideas came. Near the beginning of his career he wrote an elementary geometry in which the usual distinctions between the geometry of space and that of the plane were abandoned; the perusal of his present work leaves one with the impression that it must have been a life long regret to the author that his Euclid did not spring forth in n dimensions.

The first volume, which is an elaboration of the author's Nouveau précis d'analyse infinitésimale published in 1872, comprises in four hundred pages all the general notions of analysis from the integral number to the integrals of partial differential equations, without mention of a particular function, a straightforward, iron-heeled, uninterrupted march of general principles. The power of abstraction demanded is something terrific; the reader experiences all of the symptoms of that peculiar malady encountered by travellers at high altitudes, and if modern theories be true, some of the author's ethereal oxygen ought to be replaced by the carbon dioxide of regions nearer us common mortals. It is upon this ground that the volume cannot be judged as cordially on pedagogic as upon æsthetic principles; this criticism would be more unjust had not the author addressed his work not only to savants but primarily to students. It may be remarked here however, parenthetically, that the first volume is not half so disheartening when accompanied by the second.

But on the other hand when the work is contemplated as the consistent and logical development of an idea, as the persistent adherence to a method until it had mastered the difficulties interposed by its critics, as the epic of a human ideal, Méray's work is a triumphant success. No student who has had sufficient discipline to maintain him in its perusal can read the work without being profoundly grateful; no student who is endeavoring, for his own peace of mind and safety of soul, to construct his own theory of the number system and to frame his own theory of functions, can afford not to have availed himself of Méray's admirable work; many who cannot accept all of his dogmas—and dogmas are as deplorable in science as in religion—will recall with regret that they never sat under the author's tutelage.

A detailed analysis of this extensive treatise would ten times exceed the space that the BULLETIN can place at the disposal of the reviewer; in order to respect this restriction of space and at the same time acquaint the reader with the gist of Méray's method, the writer undertakes a succinct account of the author's memoir* on the theory of functions to which reference has already been made, noting what modifications or additions have been made in the treatise.

In the first chapter of the memoir, and the first three chapters of the treatise, Méray constructs the ensemble of fictitious numbers employed by analysis, without the aid of any other notions than those of the integral number and of the addition of such numbers. He banishes all consideration of arguments from his study of imaginaries. His notion of variant (almost identical with that of infinite rational sequence), employed to give precision to the most difficult concept of analysis, makes his exposition of the irrational both clear and luminous. In the preface to the first volume of the treatise, Méray substantiates the claim made by his friends that the latter solution was first given by him. The invention of this theory of incommensurables has been credited to Heine and the first use made of it to Lipschitz, Du Bois-Reymond, and G. Cantor. Heine's memoir entitled "Die Elemente der Functionenlehre" appeared in the seventy-fourth volume of *Crelle's Journal* with the date of 1872; Méray's Nouveau Précis is of the same date, 1872, but three years previously he had presented the theory at a congress of learned societies in Paris, in a note entitled "Remarques sur la nature des quantités définiies par la condition de servir de limite à des variables données" pub-

* Nouveau Précis d'Analyse infinitésimale, Paris, F. Savy, 1872; 8vo, xxiii + 310. This work is now included in the list of publications of Gauthier-Villars. The memoir was dedicated to M. Faurie, General Inspector of secondary instruction, and was seriously designed to bring about a revolution against the traditional methods of teaching the infinitesimal calculus. For convenience of comparison the order of ideas exhibited in the subjects of the chapters is here appended:—1° preliminaries; 2° generalities on series; 3° series arranged in the ascending powers of the independent variables; 4° olotropic functions, classification of their derivatives, general properties; 5° limits of convergence of the developments of olotropic functions; 6° composite functions; 7° principles of the discussion of functions; 8° various propositions on olotropic or quasi-olotropic functions of a single variable; 9° fundamental principle of the integral calculus; 10° theory of implicit functions; study of some implicit functions of a single variable defined by a unique equation; 12° analytic theory of the exponential function and of its related functions; 13° inverse functions; 14° ordinary differential equations, principal case; 15° general case of the preceding; 16° total differential equations; 17° simple integrals.

lished in the *Revue des Sociétés savantes* (*Sciences mathématiques*, vol. 4, p. 284, 1869).

The number system thus established in a manner that leaves nothing to be desired in the way of cold logic, the author proceeds to present the properties of series, and in particular those of integral power series. He loses no opportunity to impress his conviction that all analytic functions are identical with integral series. To Méray's mind no function exists which cannot be expanded by Taylor's theorem. All other functions he rules out as "phantasies of the mathematical imagination;" but does the reader clearly understand the meaning of the last phrase? To the same exile he consigns the non-euclidean geometry. The demands of analysis required that the idea of the number system be enlarged to include certain fictitious elements that could equally be dubbed fantastic creatures of a mathematical imagination; Méray fell in with the call of the hour and led the procession. For this, among other reasons, it is difficult to understand why the author refuses to respect the discoveries of the last twenty years and accord to the notion function the possibility of a similar enlargement. Méray is a pure mathematician, an analyst of power, and yet he would prescribe a definite boundary to the theory of functions and to all physical phenomena to come within the ken of men because "Tout phénomène naturel est représentable exactement par des séries entières, approximativement par leurs premiers termes, dont des observations de plus en plus précises fournissent empiriquement les coefficients dans l'ordre même où l'Analyse les range."

Lie averted the current objections to his method from being hurled against his volume on differential equations by limiting it to differential equations that admit of infinitesimal transformations; this was thoroughly scientific and did not carry with it the absurd denial of the existence of differential equations not admitting of infinitesimal point transformations, the number of which equations is legion. Similarly no complaint could be lodged against Méray had he contented himself with the study of functions developable by Taylor's theorem, without defining analysis to be the systematic study of the general properties of functions. He got hold of a large idea, and exhausted it ten, perhaps twenty, years ago; since then it seems to have been master of the man. Would that the same vigorous analytical power had been turned in the meantime on those spurned subjects.

The exposition of the general properties of integral series

is considerably simpler in the treatise than in the memoir because of the introduction of the author's new demonstration* of a lemma of Cauchy; the demonstration gives the lemma the position of a new theorem. The discussion of the continuity and convergence of series is admirable.

Passing to the notion of function, of which more or less has been remarked already, the author defines a function to be olotropic in the areas S_x, S_y, ···, with the olometers δ_x, δ_y, ···, when, for every system of numbers, x_0, y_0, ···, taken within the areas in question, we can develop the function in a convergent integral power series in $x - x_0$, $y - y_0$, ···, provided that the moduli of these differences are respectively less than δ_x, δ_y, ···. The areas S_x, S_y, ···, are any whatsoever having simple or multiple contours.

The successive derivatives of a function of one or more variables are naturally defined after Lagrange's theory of analytic functions and Arbogast's method of derivatives, the classical objections to these being avoided by considering only olotropic functions. The derivatives are obtained algebraically without reference to infinitesimals by developing the series $f(x + h, y + k, \cdots)$ and throwing the development into the form $f(x, y, \cdots) + hf_x' + kf_y' + \cdots$, the quantities f_x', f_y', ··· being convergent series.

A capital element in the generation of new functions is the operation which the author calls *cheminement;* this operation is the ensemble of the following: To obtain in X, Y, ··· the value of a function $f(x, y, \cdots)$ it is necessary to start from an initial system x_0, y_0, ···, and interpolate, if possible, between it and the proposed system X, Y, ··· intermediate systems x_1, y_1, ···; x_2, y_2, ···; ···; x_k, y_k ···, such that the differences between the corresponding values of the variables in two consecutive systems shall have moduli less than the olometers of the function for the values of the smaller index; then to calculate the function $f(x_1, y_1, \cdots)$ by Taylor's formula, by putting

$$f(x_1, y_1, \cdots) = f[x_0 + (x_1 - x_0), \quad y_0 + (y_1 - y_0), \cdots];$$

then $f(x_2, y_2, \cdots)$ in a similar manner by means of $f(x_1, y_1, \cdots)$; and finally $f(X, Y, \cdots)$ from $f(x_k, y_k, \cdots)$.

The author studies the theory of indefinite integrals in the eighth chapter of the treatise under the title inverse calculus of derivatives. Up to this point inclusive it has been a question only of isolated functions, that is to say

Bulletin des Sciences mathématiques, vol. 15 (1891).

functions considered independently, as if no other functions existed. The remainder of the volume is occupied with general operations which imply the composition of functions.

In the tenth chapter of the treatise's first volume the author demonstrates the convergence of the elementary developments of the integrals of a system of total differential equations, by a direct examination of these developments themselves and without the intervention of auxiliary differential equations. It is at this juncture that Méray introduces an auxiliary notion, attributed to Weierstrass but employed by Méray since 1868 and given at § 139 of the Nouveau Précis, and § 300 of the first volume. This auxiliary artifice he calls the *majorante* and defines it as follows: Given an olotropic function $f(x, y, \cdots)$ of any variables, we call majorante of this function at $x_0, y_0, \cdots$ every function $\varphi(x, y, \cdots)$ of the same variables, possessed of the property that, for $x = x_0, y = y_0, \cdots$, its value and that of all its derivatives shall be real and positive, and superior to the moduli of the corresponding values of $f(x, y, \cdots)$ and its derivatives.

The memoir contains practically nothing of the chapter XII. of the first volume on partial differential equations; the material is drawn largely from two papers published by the author with the collaboration of M. Riquier.*

In the last chapter of the volume on systems of total differential equations all reference to unsolved equations and to those of order higher than the first is omitted. In his Nouveau Précis the author demonstrated for the first time the possibility of resolving normally the integral equations with regard to the arbitrary constants; a demonstration neither without use nor without value.

As remarked before, the first volume contains no reference to any special form of function; the chapters of the memoir relating to particular functions were reserved for the second volume.

Among the elements of the first chapter we find the author's more recent demonstration† of the existence of roots of every integral algebraic equation; this demonstration is as simple and elegant as could be desired.

The limits of space prescribed forbid calling attention to the details of this second volume. Certain striking ones

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Annales scientifiques de l'École Normale supérieure, vol. 6 (1889), vol. 7 (1890).

† "Méthode directe fondée sur l'emploi des séries pour prouver l'existence des racines des équations entières à une inconnue par la simple exécution de leur calcul numérique," *Bulletin des Sciences mathématiques*, vol. 15 (1891).

however demand a word. In the third chapter the author develops the theory of radicals as a simple corollary of his theory of implicit functions elaborated in the first volume. He incorporates his memoir* which treats of this question in harmony with the fundamental point of view of Abel, whom Méray places with Lagrange as the first to found the infinitesimal analysis on a solid basis. The author's method is long, but he avoids all trigonometric considerations and thus adds a further illustration that the intrusion of geometric facts is not a necessity in analysis.

In the fourth chapter he reproduces and improves the method given in the Nouveau Précis for extending the capital discovery of Puiseux on algebraic irrationals to the roots of any olotropic equation. A principle is introduced under the name principle of the conservation of the number of roots which later brings into evidence certain fundamental properties of biperiodic functions; the principle is one of those which exist only by virtue of extending the ordinary algebraic operations to imaginaries. The author regrets, as does the reader, that the inflexible rigor which he has made his law would not permit of abridging these earlier chapters.

The Napierian logarithm, being the first transcendent having a derivative, naturally suggests itself as the first transcendent to be studied. The author then presents the exponential as the simple result of the analytic inversion of the logarithm. The inversion of the indefinite integral of a rational differential gives other functions, no longer indefinitely olotropic as the exponential, but indefinitely meromorphic; in this way the circular functions are introduced. Méray returns to the differential equation as the starting point for his elliptic functions, regarding his demonstration† of the non-nullity of the moment of the periods as sufficiently simple, and considering the dominant rôle assumed by the theta functions in modern expositions as artificial. As usual, it is the most general function that he studies.

As to the functions of Euler studied in the last chapter, they also have their origin in integration, not directly or indirectly from indefinite integrals considered as functions of their principal variables, but as functions of parametric variables which accompany the principal variable in certain

* "Théorie des radicaux fondée exclusivement sur les propriétés générales des séries entières," *Revue bourguignonne de l'Enseignement supérieur*, vol. 1 (1891).

† "Sur l'existence effective des deux périodes des fonctions elliptiques," *Ann. scient. de l'École Normale sup.*, 3d series, vol. 1 (1894).

artificial definite integrals; by the latter Méray means the integral of $f(x)$ from x_0 to X when the path of integration contains any value of x_1 singular for $f(x)$, or when the path is unlimited in any direction.

The course of the development of the whole treatise is admirably exhibited by the subjects and order of the chapters of the several volumes, which are appended below in conclusion.* The elegant applications presented in the last two parts are more than worthy of a direct reference, for they demonstrate again the success of the method in the hands of the author.

I. 1° Preliminary generalities comprising a review of the fictitious quantities upon which the speculations of modern analysis are founded; fractions, positive and negative quantities; 2° continuation of the preceding, variants in general; incommensurable quantities; 3° continuation of the two preceding chapters, imaginary quantities; 4° series in general; 5° integral series; 6° derivatives of olotropic† functions, usual genesis of these functions; 7° fundamental properties of functions which are olotropic within given areas; 8° inverse calculus of derivatives; 9° composite functions; 10° essential principle of the theory of total differential equations; 11° implicit functions in general; 12° essential principles of the theory of partial differential equations; 13° further study of immediate systems of total differential equations; addition, 1° on an

*The first volume contains a list of Méray's principal publications. The works have already been referred to; the memoirs are to be found in the following journals: *Nouv. Ann. de Math.*, 1st ser., vol. 13 (1854), 3d ser., vol. 8 (1889), vol. 9 (1890), vol. 11 (1892); *Comptes rendus*, vol. 40 (1855), vol. 106 (1888); *Ann. di Matem.*, 1st ser., vol. 3 (1860); *Ann. scient. de l'Ec. Norm sup.*, 1st. ser., vol. 4 (1867), 2d ser., vol. 6 (1877), vol. 8 (1879), vol. 12 (1883), 3d ser., vol. 1 (1884), vol. 2 (1885), vol. 6 (1889), vol. 7 (1890); *Revue des Sociétés savantes* (*Sc. math., phys., et nat.*), 2d ser., vol. 3 (1868), vol. 4 (1869); *Liouville's Journal*, 3d ser., vol. 10 (1884); *Bull. des Sc. math.*, 2d ser., vol. 12 (1888), vol. 5 (1891); *Revue bourguignonne de l'Enseign. sup.*, vol. 1 (1891), vol. 2 (1892); *Ann. de la Fac. des Sc. de Toulouse*, vol. 4 (1891).

†Mathematics of necessity mutilates language occasionally in the construction of its technical terms, but it is to be regretted that Méray did not give heed to a correction in the form of this term suggested by the editors of the *Bulletin des Sciences mathématiques* in 1874; orthography should respect etymology and the Greek derivative is ὅλος. It is interesting to observe that Méray used this term in 1872, but that not until 1875 did its successful rival *holomorphic* appear; see Briot and Bouquet's Theory of elliptic functions, 2nd edition, p. 14. In the preface to the first volume Méray anticipates a day when both will disappear, namely at that epoch when mathematicians shall have opened their eyes to the accuracy of the conceptions of Lagrange and have ceased to contemplate functions non-integrable, continuous but without derivatives, et cætera-

essential property of integral polynomials in a single variable.

II. 1° Olotropic functions of a single variable, in general; 2° meromorphic functions of a single variable, in general; 3° simple radical function; 4° study of the principle critical phases of an implicit function of a single variable defined by a unique equation; 5° Napierian logarithm and exponential function; 6° circular functions; 7° development of circular functions in series of simple fractions and in factorial series; 8° summary theory of elliptic functions; 9° continuation of the preceding, biperiodic functions in general; 10° continuation of the two preceding chapters, development of biperiodic functions in series of simple fractions or of circular functions, and in factorial series; 11° continuation of the three preceding, salient points of the theory of biperiodic functions of the second order; 12° continuation of the four preceding chapters, canonical elliptic functions; 13° notions on eulerian functions.

III. 1° Indefinite integration of current differentials; 2° calculation of certain definite integrals not demanding the knowledge of indefinite integrals; 3° elementary differential equations; 4° partial differential equations of the first order; 5° questions of maxima and minima; 6° real multiple integrals, additions: 1° elementary proposition to be substituted for Cauchy's lemma in the general theory of functions; 2° direct demonstration of Cauchy's lemma; 3° on the non-nullity of the differential determinant with respect to arbitrary constants of the general integrals of a system of total differential equations; 4° on the possibility of making the modulus of an indefinitely olotropic function increase without limit; 5° on an extended case in which interpolation permits of representing a function with an indefinite approximation.

IV. 1° Preliminaries, rectifications, quadratures, cubatures; 2° contacts in general; 3° contacts of surfaces and of lines with figures of the first degree; 4° enveloping figures; 5° contacts of first order between the sphere or the circle and given figures; 6° salient properties of ordinary surfaces; 7° contacts of higher orders of a line with the circle and the sphere; 8° questions relating to contacts of second order of a surface with the circle and right line; addition, principal formulæ in polar coördinates.

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NOTES.

THE first number has just appeared of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY, published by the Society with the coöperation of Harvard University, Yale University, Princeton University, Columbia University, Haverford College, Northwestern University, Cornell University, The University of California, Bryn Mawr College, The University of Chicago; edited by ELIAKIM HASTINGS MOORE, ERNEST WILLIAM BROWN, THOMAS SCOTT FISKE; New York, The Macmillan Company. The first number, which consists of 96 pages, contains the following articles: —"Conics and cubics connected with a plane cubic by certain covariant relations," by H. S. WHITE; "Formentheoretische Entwickelung der in Herrn White's Abhandlung über Curven dritter Ordnung enthaltenen Sätze," by P. GORDAN; "Sur la définition générale des fonctions analytiques, d'après Cauchy," by E. GOURSAT; "On a class of particular solutions of the problem of four bodies," by F. R. MOULTON; "Definition of the abelian, the two hypoabelian, and related linear groups as quotient groups of the groups of isomorphisms of certain elementary groups," by L. E. DICKSON; "Note on the unilateral surface of Moebius," by H. MASCHKE; "On regular singular points of linear differential equations of the second order whose coefficients are not necessarily analytic," by M. BÔCHER; "The elliptic sigma functions considered as a special case of the hyperelliptic sigma functions," by O. BOLZA; "On the groups which are the direct products of two subgroups," by G. A. MILLER; "On certain crinkly curves," by E. H. MOORE; "A new definition of the general abelian linear group," by L. E. DICKSON.

A NEW Annual Register of the AMERICAN MATHEMATICAL SOCIETY, including a list of members, directory, list of publications, constitution, by-laws, and the reports of the Treasurer and Librarian has recently been published and distributed to the members of the Society. Copies of the Register may be obtained from the Secretary.

AT the third ordinary meeting of the eighteenth session of the Edinburgh mathematical society, January 12, 1900, an address was delivered by Dr. PEDDIE on "The dissipation of energy in vibrating matter." Professor GIBSON'S recent papers on the theory of proportion will be published in full in the society's *Proceedings*.

THE Circolo matematico di Palermo elected the following council for the years 1900–1902 at its recent annual meeting :—Resident members, Professors M. L. Albeggiani, M. Gebbia, F. Gerbaldi, G. B. Guccia, G. Torelli ; Non-resident members : Professors E. Beltrami, of Rome, L. Bianchi, of Pisa, A. Capelli, of Naples, V. Cerruti, of Rome, L. Cremona, of Rome, P. Del Pezzo, of Naples, A. Del Re, of Naples, G. Loria, of Genoa, G. Mittag-Leffler, of Stockholm, E. Pascal, of Pavia, G. Peano, of Turin, S. Pincherle, of Bologna, H. Poincaré, of Paris, A. Tonelli, of Rome, V. Volterra, of Turin.

THE following papers were announced for the regular monthly meeting of the London mathematical society held January 11, 1900 :—"A problem in resonance, illustrative of the mechanical theory of selective absorption," by Professor H. LAMB, and "Elementary distributions of plane stress," by Mr. J. H. MICHELL.

At the preceding meeting of December 14, 1899, the list of papers presented was as follows :—"A method for extending the accuracy of mathematical formulæ," and "Central difference formulæ," by Mr. W. F. SHEPPARD ; "Circular cubics," by Mr. A. B. BASSETT ; "The theorem of residuation, being a general treatment of the intersections of plane curves at multiple points," by Dr. F. S. MACAULAY ; "The genesis of the double gamma functions, by Mr. E. W. BARNES ; "On the expression of spherical harmonics as fractional differential coefficients," by Mr. J. ROSE-INNES ; "Sums of greatest integers," by Dr. G. B. MATHEWS.

AT the meeting of the Royal Society of Edinburgh on December 19, 1899, Professor TAIT presented a "Note on the claim recently made for Gauss to the invention of quaternions," whose object was to show that what Professors Klein and Sommerfeld ascribed to Gauss in their treatise Ueber die Theorie des Kreisels was not the Hamiltonian quaternion, but a particular and limited kind of strain which consisted of a simple rotation combined with an isotropic expansion, and thus involving four constants only. Dr. C. G. KNOTT contributed a detailed criticism of the section devoted to the discussion of the theory of quaternions in the treatise cited.

THE annual meeting of the British mathematical association was held January 20, 1900, and the following papers were presented :—"Dynamical applications of the theory of correspondences," by Professor R. S. BALL ; "Triangles

triply in perspective," by Mr. J. A. THIRD; "The teaching of indices and surds," by Professor R. W. GENESE; "Illustrations of prismatic equations," by Mr. T. J. BROMWICH; "A note on the focoids," by Mr. R. F. DAVIS.

ON the twenty-fourth of December, 1899, the Physico-mathematical society of Kazan celebrated a jubilee in honor of the twenty-fifth year of the scientific career of Professor VASILIEV, who has been president of the society for fifteen years.

AMONG the various congresses in process of organization in connection with the approaching Paris Exposition there is to be one devoted to the history of the sciences. This congress proposes to consider the development of all branches of scientific knowledge. Professor PAUL TANNERY is president, and Dr. SICARD DE PLAUZOLES is secretary of the organizing committee. The official address of this committee is 10 boulevard Raspail, Paris.

THE last volume of the *Revue de Métaphysique et de Morale* contains an extended discussion on the axioms of geometry between Professor H. POINCARÉ and Mr. B. RUSSELL, occasioned by the essay of the latter on the foundations of geometry. The discussion is continued in the January number of the current volume. In this connection it may be added that Mr. FREDERICK PURSER contributes an elaborate defense of the Kantian theory of the axioms of geometry to the twenty-fifth number of *Hermathena*, edited by members of Trinity College, Dublin.

AS ONE of the by-products of the Encyclopædia of the mathematical sciences now in process of construction, Teubner, of Leipzig, announces the preparation of an elaborate series of text-books in the field of pure and applied mathematics, and seeks the coöperation of mathematicians in the project, whether or not they be members of the contributing staff of the Encyclopædia. The preliminary announcement is very meagre, but more precise details are promised at an early date. The following members of the editorial staff of the Encyclopædia, among others, have signified their approval of the undertaking :—Messrs. Bôcher, Brunel, Castelnuovo, Dingeldey, Enriques, Harkness, Kohn, Krazer, von Lilienthal, Mehmke, Netto, Pincherle, Segre, Seliwanoff, Simon, Stäckel, Staude, Vahlen, Voss, Wiman, Wirtinger, and Zeuthen.

PROFESSOR V. BJERKNES, of the University of Stockholm, has just issued the first volume of his "Vorlesungen über

hydrodynamische Fernkräfte nach C. A. Bjerknes' Theorie." This volume, illustrated with forty figures, gives an exposition, together with the author's original contributions, of the theories which have constituted the life work of his father Professor C. A. Bjerknes of the University of Christiania, and experiments which have brought to light the analogies between electrostatic and electrodynamic phenomena and the apparent attractions experienced by solid bodies plunged into moving liquids.

THE approaching volume of the *Jahresberichte* of the German mathematical association will contain, among others, the following memoirs:—"Numerical solution of equations," by R. HAUSSNER; "The kinetic problems of scientific technology," by K. HEUN; "Calculus of variations," by A. KNESER; "Graphic methods," by R. MEHMKE; "The theory of manifoldnesses," by A. SCHOENFLIES; "General dynamics," by PAUL STÄCKEL; "The theory of finite groups," by E. STEINITZ.

THE first volume of a French edition of TCHEBYCHEV'S collected works has just appeared. The volume, which is edited by Professors MARKOF and SONINE, contains thirty-four memoirs and a portrait of their author. A German translation of Vaseliev's memoir on the mathematical work of Tchebychef is to be published immediately by Teubner, of Leipzig.

M. HERMANN, of Paris, announces the appearance of a memoir of Professor G. HUMBERT on singular abelian functions.

PORTRAITS of the following mathematicians are to be had of B. G. Teubner, of Leipzig:—Moritz Cantor, Alfred Clebsch, Hermann Grassmann, Hermann von Helmholtz, Leopold Kronecker, Sophus Lie, Nikolaj Lobatschefskij, and P. L. Tchebychef. With the exception of that of Grassmann, which is a woodcut, all are heliogravures.

THE Brussels academy of sciences has awarded its prize of six hundred francs to Professor LÉON AUTONNE, of the University of Lyons, for his work in geometry.

PARIS ACADEMY OF SCIENCES.—Professor CHARLES MÉRAY, of Dijon, has been elected a correspondent of the section of geometry.—The nominees of the academy for the post of astronomer at the Bureau des Longitudes held by the late Professor Tisserand are MM. RADAU and BIGOURDAN.—The

Bordin prize in geometry was not awarded but M. JULES DRACH received an honorable mention. This was to have been awarded in 1898 for a successful memoir on a study of questions relative to the determination as to properties and applications of systems of orthogonal curvilinear coördinates in n variables, indicating in particular and in a manner as precise as possible the degree of generality of these systems. But one memoir, of insufficient merit, was received in 1898; accordingly the competition was held open for 1899. Three new memoirs were received by the academy, only one of which was retained. This bore the unique device "L'esprit ne peut se soumettre qu' à ce qui est esprit." The commission consisted of Professors Poincaré, Picard, Darboux, M. Lévy, and Appell.—The Francoeur prize in geometry was awarded to M. LE CORDIER, with a very honorable mention to M. LE ROY.—The Poncelet prize was given to M. COSSERAT for the whole of his contributions to geometry and mechanics.—The extraordinary prize in mechanics, six thousand francs, was assigned to M. BAILLES for his treatise on the geometry of indicator diagrams, supplementary prizes being given to MM. CHARBONNIER, GALY-ACHÉ and PERRIN.—The Montyon prize was awarded to M. PARTIOT, the PLUMEY prize to M. BONJOUR, and the Fourneyron prize to M. A. RATEAU.—The Lalande prize in astronomy was awarded to Mr. W. R. BROOKS for his discoveries in connection with comets, and the Valz prize to M. NYRÉN, of Pulkowa, for his work in sidereal astronomy.—Of the general prizes, the Arago medal was conferred on Professor G. G. STOKES on the occasion of his jubilee at Cambridge, and the Petit D'Ormoy prize in mathematics was given to M. MOUTARD.—In 1900 the subject for the Grand Prize in the mathematical sciences is "to perfect in some important point the investigation of the number of classes of quadratic forms having integral coefficients in two indeterminates." The subject of the Bordin prize for the same year is "to develop and perfect the theory of surfaces applicable on a surface of revolution." The Francoeur, Poncelet, Montyon, Lalande, and Valz prizes will be awarded the same year. The *Comptes rendus* for December 18, 1899, contains full details relative to all the prizes of the academy.

UNIVERSITY OF CALIFORNIA.—The following courses in mathematics are announced for the second term of the present session:—By Professor M. W. HASKELL: Selected topics in higher mathematics, two hours; Theory of algebraic forms, three hours; Spherical harmonics, two hours; Seminar, three hours.—By Professor G. C. EDWARDS: Differential

equations, three hours.—By Mr. A. B. PIERCE: Analytical projective geometry, three hours.—By Dr. E. J. WILCZYNSKI: The hypergeometric function, two hours; Calculus of variations, one hour.—By Mr. A. W. WHITNEY: Theory of functions of a complex variable, three hours.—By Mr. N. L. PERRY: Determinants and theory of equations, two hours.

THE department of mathematics of the University of Pennsylvania announces a course of twelve lectures on space analysis to be given by Dr. ALEXANDER MACFARLANE during the month of February, 1900.

PHYSICS was the most favored subject, and mixed mathematics rather more popular than pure mathematics, with the successful candidates for the B.Sc. degree at the recent examinations of London University.

THE public discussion of the proposal to remodel the mathematical tripos at Cambridge University (see BULLETIN, 2d series, vol. 6, No. 4, January, 1900, p. 168) was, on the whole, favorable to the scheme; though the senior wrangler naturally found more than one advocate for his continued existence. Generally speaking, the younger mathematicians, many of whom had themselves headed the tripos, were urgent that the present system is prejudicial to sound mathematical learning, and, in particular, to mathematical research. The older members of the university and certain of the private tutors, while more or less willingly admitting that the system produced imperfect results, denied that the abolition of the order of merit was the proper remedy. They would prefer to see older and more experienced examiners appointed, together with some rearrangement of the marks assigned to the harder questions, without indicating precisely how these suggestions would meet the difficulties of the case. The question is exciting keen discussion in private, inasmuch as the title of senior wrangler is interwoven with the texture of Cambridge tradition. As regards titles it has been suggested that the first Smith's prizeman might bear the title of first wrangler; this would have the advantage of finality, since there is no further examination to displace the candidate. The question comes up for decision in the next Cambridge term.

THE Council of the Royal astronomical society has awarded the society's gold medal of the year 1899 to Professor H. POINCARÉ for his researches in celestial mechanics.

PROFESSORS G. DARBOUX and H. MOISSAN are the delegates of the Paris academy of sciences to the approaching

celebration of the second centenary of the Berlin academy of sciences.

PROFESSOR G. G. STOKES, of Cambridge University, has been elected a foreign associate of the section of mathematics and physics of the Brussels academy of sciences.

PROFESSOR W. BURNSIDE has been made an honorary fellow of Pembroke College, of Cambridge University.

MR. B. H. CRENSHAW has been promoted to an assistant professorship of mathematics at the Alabama Polytechnic Institute.

MR. N. L. PERRY has been appointed instructor in mathematics at the University of California. Dr. J. V. WESTFALL and Mr. AUGUST VON ENDE have been made instructors in mathematics at the University of Iowa. Mr. F. E. ROSS, graduate student of mathematics at the University of California, has been appointed instructor in mathematics at the University of Nebraska.

DR. JOHN EUGENE DAVIES, professor of mathematical physics in the University of Wisconsin, died at Chicago, January 23, 1900.

THE deaths are announced of Mr. FRANCIS GUTHRIE, professor of mathematics in South African College, aged sixty-eight years, and of Judge J. B. STALLO, of Cincinnati, Ohio, a well known patron of philosophy and mathematics, aged seventy-six years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ALASIA (G.). Geometria e trigonometria della sfera. Milano, Hoepli, 1900. 16mo. 6 + 207 pp. (Manuali Hoepli.)

BOLYAI. Wolfgangi Bolyai de Bolya Tentamen iuventutem studiosam in elementa matheseos purae elementaris ac sublimioris methodo intuitiva evidentiaque huic propria introducendi, cum appendice triplici. Editio secunda. Tomus I: Conspectus arithmeticae generalis. Mandato academiae scientiarum hungaricae suis adnotationibus adiectis ediderunt Iulius König et Mauritius Réthy, academiae scientiarum hungaricae sodales. Leipzig, Teubner, 1899. 4to. 12 + 679 pp., 11 plates, 1 portrait. Half leather. M. 40.00

BONNEL (J. F.). Les atomes et hypothèses dans la géométrie. 3e édition, revue et augmentée. Paris, Hermann, 1899. 8vo. 200 pp. Fr. 5.00

BOYER (J.). Histoire des mathématiques. Paris, Carré et Naud, 1899. 8vo. 226 pp., boards. (Bibliothèque de la *Revue générale des sciences.*) Fr. 5.00

FATTOR (L.). Sulle serie del Fibonacci; nota. Venezia, Visentini, 1899. 8vo. 10 pp.

FORMULAIRE de mathématiques publié par la *Revue de mathématiques* (*Rivista di matematica*). Directeur: G. Peano. Tome II, No. 3: Logique mathématique; arithmétique; limites; nombres complexes; vecteurs; dérivées; intégrales. Torino, Bocca, 1899. 8vo. 200 pp.

JUNG (H.). Ueber die kleinste Kugel, die eine räumliche Figur einschliesst. (Diss.) Marburg, 1899. 8vo. 26 pp.

KIEPERT (L.). Grundriss der Differential- und Integral-Rechnung. Teil II: Integral-Rechnung. 7te Auflage des gleichnamigen Leitfadens von weiland M. Stegemann. Hannover, Helwing, 1899. 8vo. 20 + 617 pp. M. 11.50

KLUSSMANN (R.). Systematisches Verzeichnis der Abhandlungen, welche in den Schulschriften sämtlicher an dem Programmtausche teilnehmenden Lehranstalten erschienen sind. Nebst zwei Registern. Vol. III: 1891-1895. Leipzig, Teubner, 1899. 8vo. 8 + 342 pp. M. 5.00

KÖNIG (J.). See BOLYAI.

LEGENDRE (A. M.). Théorie des nombres. Réimpression fac-simile de la troisième édition (1830). 2 volumes. Paris, Hermann, 1899. 4to. 800 pp. Fr. 40.00

MILLER (G. A.). On several classes of simple groups. (*Proceedings of the London Mathematical Society*, Vol. 31, No. 688.) 8vo. Pp. 148–150.

PEANO (G.). See FORMULAIRE.

PRZEWALSKI (I.). Analytic geometry, with a collection of problems. 4th edition. Moscow, 1899. 8vo. 310 pp. (Russian.) M. 7.00

RÉTHY (M.). See BOLYAI.

RICHARD (J.). Leçons sur les méthodes de la géométrie moderne. Paris, Hermann, 1899. 8vo. 210 pp. Fr. 6.00

STEGEMANN (M.). See KIEPERT (L.).

VECCHI (M.). Sulla funzione ζ di Riemann; gli zeri della funzione ζ. Parigi, Hermann, 1899. Fr. 1.50

II. ELEMENTARY MATHEMATICS.

ANTOLINI (G.). Nozioni elementari di algebra per le scuole tecnichee per la prima classe delle scuole normali conforme agli ultimi programmi ministeriali. Empoli, Traversari, 1899. 16mo. 57 pp. Fr. 0.90

ARBUZOV (V.), MININ (A.), MININ (V.), and NAZAROV (D.). Systematic collection of problems in arithmetic, for use in gymnasia, seminaries, etc. 8th edition. Moscow, 1899. 8vo. 244 pp. (Russian.) R. 0.75

BAUR (L.). Sammlung arithmetischer Aufgaben. Durch Erläuterungen, Andeutungen und vollständig ausgeführte Auflösungen eingerichtet für Lehrer und Lehramts-Kandidaten (Reallehrer, Kollaboratoren, Volksschullehrer etc.), sowie zum Selbststudium. Horb, Christian, 1899. 8vo. 8 + 211 pp. Cloth. M. 3.00

BENOIT (P.). Essai d'une géométrie nouvelle. Saint-Dié, Weick [1899]. 8vo. 153 pp.

BERTRAND (J.) et GARCET (H.). Traité d'algèbre. Première partie, à l'usage des classes de mathématiques élémentaires. 17e édition. Paris, Hachette, 1899. 8vo. 4 + 330 pp. Fr. 5.00

BLUMBERG (I.). Complementary algebra, preceded by a paper on the application of algebra to geometry. 6th edition, revised. St. Petersburg, 1899. 8vo. 148 pp. (Russian.) R. 1.00

CHEVREL. See FITZ-PATRICK.

EICHHORN (W.). Arithmetisches Regelheft nebst Wiederholungstafeln. Mit einer Begleitschrift als Vorwort und Gebrauchsanweisung. Als Ergänzung zu einem jeden Lehrbuche der Arithmetik zusammengestellt. In 4 Heften. (1. Quarta (Quinta): Rechnen als Vorstufe der Arithmetik. 40 pp. M. 0.40.—2. Untertertia: Grundrechnungsarten mit allgemeinen Zahlen; Gleichungen. 32 pp. M. 0.40 — 3. Obertertia: Proportionen, Potenzen, Wurzeln, Gleichungen. 42 pp. M. 0.40.—4. Untersekunda: Logarithmen, Reihen, Zinseszins- und Renteurechnung. 23 pp. M. 0.30.) Leipzig, Teubner, 1899. 8vo.

ESTEL (V.). See SÄRCHINGER (E.).

F. (F.). Solutions des exercices du Cours élémentaire de géométrie. Paris, Poussielgue [1899.] 16mo. 81 pp. (Collection d'ouvrages classiques rédigés en cours gradués, conformément aux programmes officiels.)

FITZ-PATRICK ET CHEVREL. Exercices d'arithmétique, énoncés et solutions, avec une préface de J. Tannery. Deuxième édition, augmentée des énoncés de 800 problèmes d'arithmétique théorique et commerciale, d'un résumé et exercices d'arithmétique commerciale. Paris, Hermann, 1899. 8vo. 690 pp. Fr. 10.00

FURST (S. W.). Mensuration, with special application of the prismoidal formula. Harrisburg, Pa., Myers, 1899. 16mo. 72 pp. Cloth. $0.50

GARCET (H.). See BERTRAND (J.).

GLÖSER (M.). Grundzüge der allgemeinen Arithmetik für die dritte Classe der österreichischen Realschulen. 4te Auflage. Wien, Pichler, 1899. 8vo. 3 + 116 pp. Cloth. M. 1.30

——. Lehrbuch der Arithmetik für die erste und zweite Classe der österreichischen Realschulen. 4te Auflage. Wien, Pichler, 1899. 8vo. 5 + 209 pp. Cloth. M. 1.80

KLAS (A.). Die Dreiteilung und Fünfteilung des Winkels auf dem Wege der elementaren Geometrie, allein mit Lineal und Zirkel gelöst und dargelegt. Wiesbaden, Ferger, 1899. 4to. 14 pp., 8 plates. M. 1.20

LAZZARI (G.). Manuale di trigonometria piana. Livorno, Giusti, 1899. 16mo. 4 + 124 pp. (Biblioteca degli studenti; riassunti per tutte le materie d'esame nei licei, ginnasi, istituti tecnici, ecc.) Fr. 1.00

LOEWENSTERN. Answers to the questions in theoretical arithmetic set at the entrance examinations of the Polytechnic School. Moscow, 1899. 8vo. 46 pp. (Russian.) R. 0.75

MILNE (W. J.). Key to Milne's Plane and solid geometry. New York, American Book Co. [1899.] 12mo. 313 pp. Half leather. $1.25

MININ. See ARBUZOV (V.).

MOLA (P.). Appunti di trigonometria. Torino, Prete, 1899. 16mo. 33 pp., 2 plates. (Collana scientifica.)

MURRAY (D. A.). Plane trigonometry; for colleges and secondary schools. New York, Longmans, Green & Co., 1899. 12mo. 13 + 206 pp. Cloth. $0.90

NAZAROV (D.). See ARBUZOV (V.).

NIKULTSEV (P.). Course in arithmetic for secondary schools. 5th edition. Moscow, 1900. 8vo. 277 pp. (Russian.) R. 1.00

PINCHERLE (S.). Geometria metrica e trigonometria. 5a edizione. Milano, Hoepli, 1899. 16mo. 4 + 158 pp. (Manuali Hoepli.)

REINBECK. Die planimetrische Aufgabe für Quarta und Untertertia des Realgymnasiums. Uelzen, 1899. 8vo. 12 + 80 pp. M. 2.00

ROBINSON. New higher arithmetic, for high-schools, academies, and mercantile colleges. New York, American Book Company [1899]. 8vo. 527 pp. Half leather. $1.00

SÄRCHINGER (E.) und ESTEL (V.). Resultate zur zweiten Auflage der Aufgabensammlung für den Rechenunterricht in den Unterklassen der Gymnasien. 2tes Heft: Quinta. 58 pp. M. 1.50.—3tes Heft: Quarta. 26 pp. M. 0.70. Leipzig, Teubner, 1899. 8vo. Boards.

SASSENFELD (I.). Die Hauptsätze der Elementar-Mathematik für das Gymnasium. Nach den neuen Lehrplänen bearbeitet. In 6 Lehrgängen. (1. Planimetrie. (Quarta.) 20 pp. 2. Planimetrie. (Untertertia.) 20 pp. 3. Planimetrie. (Obertertia.) 20 pp. 4.

Planimetrie; Einleitung in die Trigonometrie und Stereometrie; Anhang: Die Logarithmen. (Untersekunda.) 24 pp. 5 and 6. Abschluss der Planimetrie, Trigonometrie und Stereometrie; Einleitung in die analytische Geometrie der Ebene. (Obersekunda und Prima.) 64 pp.) Trier, Lõwenberg, 1899. 8vo. M. 3.50

——. Lehr- und Uebungsbuch der Arithmetik und Algebra für das Gymnasium. Trier, Lõwenberg, 1899. 8vo. 7 + 111 pp. M. 1.80

SHMULEVICH (P.). Collection of 1220 problems in algebra and arithmetic, with complete solutions. 2d edition. St. Petersburg, 1900. 8vo. 466 pp. (Russian.)

TANNERY (J.). See FITZ-PATRICK.

TESTI (G. M.). Corso di matematiche ad uso delle scuole secondarie e più specialmente degli istituti tecnici. Vol. II: Algebra elementare. 2a edizione, intieramente rifatta. Livorno, Giusti, 1900. 8vo. 399 pp. Fr. 2.80

——. Elementi di matematica ad uso degli alunni delle scuole normali. Fasc. 1: Nozioni di algebra, radici quadrate e cubiche. Per la prima classe normale. 3a edizione. Livorno, Giusti, 1899. 16mo. 8 + 88 pp. Fr. 0.80

——. Fasc. 2: Geometria piana; generalità, eguaglianza, equivalenza e misura. Per la prima classe normale. 3a edizione. Livorno, Giusti, 1899. 16mo. 7 + 78 pp. Fr. 0.80

——. Fasc. 3: Aritmetica razionale; norme didattiche per l'insegnamento dell' aritmetica nelle scuole elementari inferiori. Per la seconda classe normale. 3a edizione. Livorno, Giusti, 1899. 16mo. 7 + 105 pp. Fr. 0.90

——. Fasc. 4: Proporzionalità o similitudine; norme per l'insegnamento della geometria piana nelle scuole elementari. Per la seconda classe normale. 3a edizione. Livorno, Giusti, 1899. 16mo. 60 pp. Fr. 0.60

——. Fasc. 5: Proporzionalità e problemi relativi; norme didattiche per l'insegnamento dell' aritmetica nelle scuole elementari superiori. Per la terza classe normale. 3a edizione. Livorno, Giusti, 1899. 16mo. 7 + 76 pp. Fr. 0.70

——. Fasc. 6: Geometria solida; norme didattiche per l'insegnamento della geometria solida nelle scuole elementari. Per la terza classe normale. 3a edizione. Livorno, Giusti, 1899. 16mo. 99 pp. Fr. 0 90

WARREN (I.). Elements of plane trigonometry; for schools and universities. 7th edition. London, Simpkin, 1899. 12mo. 194 pp. 3s. 6d.

WROBLEWSKI (W.). Solutions of the collection of exercises in F. Bychkov's Course of elementary algebra. Part IV: Quadratic and cubic equations. St. Petersburg, 1900. 8vo. 286 pp. (Russian.) R. 2.00

III. APPLIED MATHEMATICS.

AIMÉ (G.). See DUHEM (P.).

BARR (J. H.). Kinematics of machinery; a brief treatise on constrained motions of machine elements. New York, Wiley, 1899. 8vo. 5+247 pp. Cloth. $2.50

BILFINGER (G.). Untersuchungen über die Zeitrechnung der alten Germanen. Teil I: Das altnordische Jahr. Stuttgart, 1899. 8vo. M. 2.50

DUHEM (P.). La tension de dissociation avant H. Saint-Claire Deville.—De l'influence de la pression sur les actions chimiques par G. Aimé (1837), avec une introduction par P. Duhem. Paris, Hermann, 1899. 8vo. Fr. 2.00

EWERS (P.). Zur Mechanik der Kanal- und Kathodenstrahlen. (Diss.) München, 1899. 8vo. 52 pp.

FALLER (O.). Eine neue Anschauung über die Reibung. Vorläufige Mitteilung. Vortrag. München, Ackermann, 1899. 8vo. 16 pp. M. 0.40

GERHARDT (F.). Ueber den Gang des Dampfdrucks in den wärmsten Tagesstunden. (Diss.) Berlin, 1899. 8vo. 35 pp.

GRAWINKEL (C.). See THOMPSON (S. P.).

HUMMEL (L.). See REBBER (W.).

LEHFELD (R. A.). Text-book of physical chemistry. London, Arnold, 1899. 12mo. 320 pp. 7s. 6d.

MAFFIOTTI (G. B.). Il planimetro a scure di H. Prytz; teoria e pratica. Torino, Paravia, 1899. 8vo. 31 pp. Fr. 1.60

NOWAK (A.). Beispiele aus der Festigkeitslehre. 2te Auflage. Mittweida, 1899. 8vo. 38 pp., 1 plate. Boards. M. 1.50

PIGNATARI (G.). Piani ed elissi centrali nei sistemi di forma invariabile. Napoli, Priore, 1899. 8vo. 11 pp.

REBBER (W.). Die Festigkeitslehre und ihre Anwendung auf den Maschinenbau. Elementar behandelt zum Gebrauche für Studierende und in der Praxis. 4te Auflage, herausgegeben von L. Hummel. Mittweida, Polytechnische Buchhandlung, 1899. 8vo. 8 + 584 pp. Cloth. M. 12.00

RITTER (A.). Lehrbuch der technischen Mechanik. 8te Auflage. Leipzig, Baumgärtner, 1899. 8vo. 15 + 801 pp. M. 20.0

SAILER (E.). Die Aufgaben aus der darstellenden Geometrie, welche bei der Prüfung für das Lehramt der Mathematik und Physik an den königl. bayerischen humanistischen und technischen Unterrichts-Anstalten in den Jahren 1873 bis 1893 gestellt wurden. München, Ackermann, 1899. 8vo. 75 pp. M. 2.00

SCHLOTKE (J.). Lehrbuch der darstellenden Geometrie. Teil 2: Schatten- und Beleuchtungslehre. 2te Auflage. Dresden, Kühtmann, 1899. 8vo. 4 + 60 pp. M. 2.00

SHULGIN (G.). Nautical astronomy. St. Petersburg, 1899. 8vo 520 pp., 2 maps. (Russian.)

STRECKER (K.). See THOMPSON (S. P.).

THOMPSON (S. P.). Die dynamoelektrischen Maschinen. 6te Auflage. Nach. C. Grawinkel's Uebersetzung neu bearbeitet von K. Strecker und F. Vesper. Teil I. Halle, Knapp, 1899. 8vo. 11 + 374 pp., 12 plates. M. 12.00

VESPER (F.). See THOMPSON (S. P.).